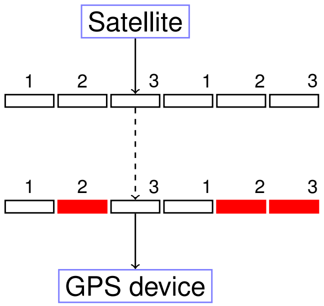
Outline

- Erasure Codes
- Error Correction
- More Polynomials!

Erasure Codes.



3 packet message. So send 6!

Lose 3 out 6 packets.

Gets packets 1,1,and 3.

Problem: Want to send a message with n packets.

Channel: Lossy channel: loses k packets.

Question: Can you send $n+k$ packets and recover message?

Solution Idea: Use Polynomials!!!

Solution Idea.

n packet message, channel that loses k packets.

Must send at least $n+k$ packets!

Any n packets should allow reconstruction of n packet message.

Any n point values allow reconstruction of degree $n-1$ polynomial which has n coefficients!

We have a strategy!

Use polynomials.

Problem: Want to send a message with n packets.

Channel: Lossy channel: loses k packets.

Question: Can you send $n+k$ packets and recover message?

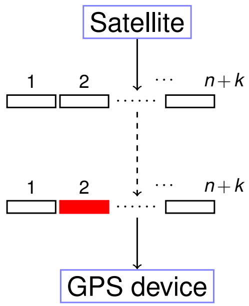
A degree $n-1$ polynomial determined by any n points!

Erasure Coding Scheme: Message = $m_0, m_1, m_2, \ldots, m_{n-1}$.
Each m_i is a packet.

1. Choose prime $p > 2^b$ for packet size b (size = number of bits).
2. $P(x) = m_{n-1}x^{n-1} + \cdots m_0 \pmod p$.
   Each $m_i \in \{0, 1, \ldots, p-1\}$
3. Send $P(1), \ldots, P(n+k)$. ($p > n+k$)

Any n of the $n+k$ packets gives polynomial ...and message!

Erasure Codes.



n packet message. So send $n+k$!

Lose k packets.

Any n packets is enough!

n packet message.

Optimal.

Comparison with Secret Sharing.

Comparing information content:

Secret Sharing: each share is size of whole secret.

Coding: Each packet has size $1/n$ of the whole message.

Erasure Code: Example.

Send message of 1,4, and 4. up to 3 erasures. $n = 3, k = 3$

Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$.

How?

Lagrange Interpolation. (sum of Δ_i polynomials)
Linear System of Equations. (in modular arithmetic)

Suppose we work modulo 5.

$P(x) = x^2 \pmod 5$
$P(1) = 1, P(2) = 4, P(3) = 9 = 4 \pmod 5$

Send $n + k = 6$ packets: $(0, P(0)) \ldots (5, P(5))$.

6 points. Better work modulo 7 at least!

Why? $(0, P(0)) = (5, P(5)) \pmod 5$

Example

Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$.

Modulo 7 to accommodate at least $n + k = 6$ packets.

Linear equations:

$$\begin{aligned} P(1) = a_2 + a_1 + a_0 &\equiv 1 \pmod 7 \\ P(2) = 4a_2 + 2a_1 + a_0 &\equiv 4 \pmod 7 \\ P(3) = 2a_2 + 3a_1 + a_0 &\equiv 4 \pmod 7 \end{aligned}$$

$6a_1 + 3a_0 = 2 \pmod 7$, $5a_1 + 4a_0 = 0 \pmod 7$

$a_1 = 2a_0$. $a_0 = 2 \pmod 7$ $a_1 = 4 \pmod 7$ $a_2 = 2 \pmod 7$

$P(x) = 2x^2 + 4x + 2$

$P(1) = 1$, $P(2) = 4$, and $P(3) = 4$

Send
Packets: $(1,1), (2,4), (3,4), (4,7), (5,2), (6,0)$

Notice that packets are of the form x, y: contain "x-values".

Check Your Understanding

You want to send a message consisting of packets 1,4,2,3,0

through a noisy channel that loses 3 packets.

How big should modulus be?
Larger than 8 $(n + k)$ and prime!

The other constraint: arithmetic system can represent $0, 1, 2, 3, 4$.

Send n packets b-bit packets, with k errors.
Modulus should be larger than $n + k$ and also larger than 2^b.

Let's Reflect: Polynomials are useful!

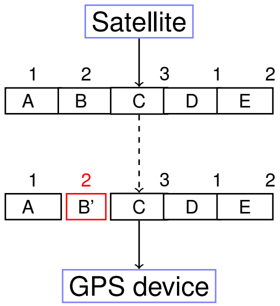
- Give Secret Sharing:
  Evaluate at $\geq k$ points to recover secret
- Give Erasure Codes:
  Send $n + k$ pairs (x, y) to reconstruct n-packet message

Next: Error Correction

Noisy Channel: corrupts k packets. (rather than loss/erasures.)

Additional Challenge: Finding which packets are corrupt.

Error Correction



3 packet message. Send 5.

Corrupts 1 packets.

The Scheme.

Problem: Communicate n packets $m_1,\ldots,m_n$ on noisy channel that corrupts $\leq k$ packets.

Reed-Solomon Code:

1. Make a polynomial, $P(x)$ of degree $n-1$, that encodes message.
   - $P(1) = m_1,\ldots,P(n) = m_n$.
   - Recall: could encode with packets as coefficients.
2. Send $P(1),\ldots,P(n+2k)$.

After noisy channel: Receive values $R(1),\ldots,R(n+2k)$.

Properties:
(1) $P(i) = R(i)$ for at least $n+k$ points i,
(2) $P(x)$ is unique degree $n-1$ polynomial that contains $\geq n+k$ received points.

Properties: proof.

$P(x)$: degree $n-1$ polynomial.
Send $P(1),\ldots,P(n+2k)$
Receive $R(1),\ldots,R(n+2k)$
At most k i's where $P(i) \neq R(i)$.

Properties:
(1) $P(i) = R(i)$ for at least $n+k$ points i,
(2) $P(x)$ is unique degree $n-1$ polynomial that contains $\geq n+k$ received points.

Proof:
(1) Easy. Only k corruptions (by assumption).
(2) Degree $n-1$ polynomial $Q(x)$ consistent with $n+k$ points.
$Q(x)$ agrees with $R(i)$, $n+k$ times.
$P(x)$ agrees with $R(i)$, $n+k$ times. (possibly different $n+k$ from above?)

Total points contained by both: $2n+2k$. P Pigeons.
Total points to choose from : $n+2k$. H Holes.
Points contained by both : $\geq n$. $\geq P-H$ Collisions.
$\implies Q(i) = P(i)$ at n points.
$\implies Q(x) = P(x)$. □

Example.

Message: $3,0,6$.

Reed Solomon Code: $P(x) = x^2+x+1 \pmod 7$ has $P(1) = 3, P(2) = 0, P(3) = 6$ modulo 7.

Send: $P(1) = 3, P(2) = 0, P(3) = 6, P(4) = 0, P(5) = 3$.

Receive $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$.

$P(i) = R(i)$ for $n+k = 3+1 = 4$ points.

Slow solution.

Brute Force:
For each subset of $n+k$ points (out of $n+2k$)
Fit degree $n-1$ polynomial, $Q(x)$, to n of them.
Check if consistent with all $n+k$ of the points.
If yes, output $Q(x)$.

- For subset of $n+k$ pts where $R(i) = P(i)$, method will reconstruct $P(x)$!
- Recall: For any subset of $n+k$ pts,
  1. there is unique degree $n-1$ polynomial $Q(x)$ that fits n of them
  2. and where $Q(x)$ is consistent with $n+k$ points
     $\implies P(x) = Q(x)$.

Reconstructs $P(x)$ and only $P(x)$!!

Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Find $P(x) = p_2x^2+p_1x+p_0$ that contains $n+k = 3+1$ points.

All equations..

$$p_2+p_1+p_0 \equiv 3 \pmod 7$$
$$4p_2+2p_1+p_0 \equiv 1 \pmod 7$$
$$2p_2+3p_1+p_0 \equiv 6 \pmod 7$$
$$2p_2+4p_1+p_0 \equiv 0 \pmod 7$$
$$1p_2+5p_1+p_0 \equiv 3 \pmod 7$$

Assume point 1 is wrong and solve..no consistent solution!
Assume point 2 is wrong and solve...consistent solution!

In general..

$P(x) = p_{n-1}x^{n-1} + \cdots p_0$ and receive $R(1), \ldots R(m = n+2k)$.

$$\begin{aligned} p_{n-1} + \cdots p_0 &\equiv R(1) \pmod{p} \\ p_{n-1}2^{n-1} + \cdots p_0 &\equiv R(2) \pmod{p} \\ &\vdots \\ p_{n-1}i^{n-1} + \cdots p_0 &\equiv R(i) \pmod{p} \\ &\vdots \\ p_{n-1}(m)^{n-1} + \cdots p_0 &\equiv R(m) \pmod{p} \end{aligned}$$

Error!! .... Where???
Could be anywhere!!! ...so try everywhere.
Runtime: $\binom{n+2k}{k}$ possibilitities.

Something like $(n/k)^k$ ...Exponential in k!.

How do we find where the bad packets are efficiently?!?!?!

Where can the bad packets be?

$$\begin{aligned} E(1)(p_{n-1} + \cdots p_0) &\equiv R(1)E(1) \pmod{p} \\ 0 \times E(2)(p_{n-1}2^{n-1} + \cdots p_0) &\equiv R(2)E(2) \pmod{p} \\ &\vdots \\ E(m)(p_{n-1}(m)^{n-1} + \cdots p_0) &\equiv R(n+2k)E(m) \pmod{p} \end{aligned}$$

Idea: Multiply equation i by 0 if and only if $P(i) \neq R(i)$.
All equations satisfied!!!!!

But which equations should we multiply by 0?

We will use a polynomial!!! One that we don't know... But can find!

Errors at points $e_1, \ldots, e_k$. (In diagram above, $e_1 = 2$.)

Error locator polynomial: $E(x) = (x - e_1)(x - e_2) \ldots (x - e_k)$.

$E(i) = 0$ if and only if $e_j = i$ for some j

Multiply equations by $E(\cdot)$. (For our example, E(x) = (x-2).)

All equations satisfied!!

Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Find $P(x) = p_2x^2 + p_1x + p_0$ that contains $n + k = 3 + 1$ points.

Plugin points...

$$\begin{aligned} (1-2)(p_2 + p_1 + p_0) &\equiv (3)(1-2) \pmod{7} \\ (2-2)(4p_2 + 2p_1 + p_0) &\equiv (1)(2-2) \pmod{7} \\ (3-2)(2p_2 + 3p_1 + p_0) &\equiv (6)(3-2) \pmod{7} \\ (4-2)(2p_2 + 4p_1 + p_0) &\equiv (0)(4-2) \pmod{7} \\ (5-2)(4p_2 + 5p_1 + p_0) &\equiv (3)(5-2) \pmod{7} \end{aligned}$$

Error locator polynomial: $(x-2)$.

Multiply equation i by $(i-2)$. All equations satisfied!

But don't know error locator polynomial! Do know form: $(x-e)$.

4 unknowns (p_0, p_1, p_2 and e), 5 nonlinear equations.

The General Case.

$$\begin{aligned} E(1)(p_{n-1} + \cdots p_0) &\equiv R(1)E(1) \pmod{p} \\ &\vdots \\ E(i)(p_{n-1}i^{n-1} + \cdots p_0) &\equiv R(i)E(i) \pmod{p} \\ &\vdots \\ E(m)(p_{n-1}m^{n-1} + \cdots p_0) &\equiv R(m)E(m) \pmod{p} \end{aligned}$$

$P(x) = p_{n-1}x^{n-1} + p_{n-2}x^{n-2} + \ldots + p_0$

$m = n + 2k$ satisfied equations, $n + k$ unknowns. But nonlinear!

Let $Q(x) = E(x)P(x) = a_{n+k-1}x^{n+k-1} + \cdots a_0$.

Rewrite the ith equation, for all i, as:

$$Q(i) = R(i)E(i) \pmod{p}.$$

Note: this is linear in a_i and coefficients of $E(x)$!

Finding $Q(x)$ and $E(x)$?

- $E(x)$ has degree k ...

$$E(x) = x^k + b_{k-1}x^{k-1} \cdots b_0.$$

- $Q(x) = P(x)E(x)$ has degree $n + k - 1$ ...

$$Q(x) = a_{n+k-1}x^{n+k-1} + a_{n+k-2}x^{n+k-2} + \cdots a_0$$

Solving for $Q(x)$ and $E(x)$...and $P(x)$

For all points $1, \ldots, i, n+2k$,

$$Q(i) = R(i)E(i) \pmod{p}$$

Gives $n + 2k$ linear equations.

$$\begin{aligned} a_{n+k-1} + \ldots a_0 &\equiv R(1)(1 + b_{k-1} \cdots b_0) \pmod{p} \\ a_{n+k-1}(2)^{n+k-1} + \ldots a_0 &\equiv R(2)((2)^k + b_{k-1}(2)^{k-1} \cdots b_0) \pmod{p} \\ &\vdots \\ a_{n+k-1}(m)^{n+k-1} + \ldots a_0 &\equiv R(m)((m)^k + b_{k-1}(m)^{k-1} \cdots b_0) \pmod{p} \end{aligned}$$

..and $n + 2k$ unknown coefficients of $Q(x)$ and $E(x)$!

Solve for coefficients of $Q(x)$ and $E(x)$.

Once we have those, compute $P(x)$ as $Q(x)/E(x)$.

Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

$Q(x) = E(x)P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

$E(x) = x - b_0$

$Q(i) = R(i)E(i)$.

$$\begin{aligned}
a_3 + a_2 + a_1 + a_0 &\equiv 3(1 - b_0) \pmod 7\\
a_3 + 4a_2 + 2a_1 + a_0 &\equiv 1(2 - b_0) \pmod 7\\
6a_3 + 2a_2 + 3a_1 + a_0 &\equiv 6(3 - b_0) \pmod 7\\
a_3 + 2a_2 + 4a_1 + a_0 &\equiv 0(4 - b_0) \pmod 7\\
6a_3 + 4a_2 + 5a_1 + a_0 &\equiv 3(5 - b_0) \pmod 7
\end{aligned}$$

$a_3 = 1$, $a_2 = 6$, $a_1 = 6$, $a_0 = 5$ and $b_0 = 2$.

$Q(x) = x^3 + 6x^2 + 6x + 5$.

$E(x) = x - 2$.

Example: Compute $P(x)$.

$Q(x) = x^3 + 6x^2 + 6x + 5$.
$E(x) = x - 2$.

```
                  1 x^2 + 1 x + 1
              -----------------
x - 2 ) x^3  + 6 x^2 + 6 x + 5
        x^3  - 2 x^2
        ----------
                  1 x^2 + 6 x + 5
                  1 x^2 - 2 x
                  ---------------
                            x + 5
                            x - 2
                            -----
                                0
```

$P(x) = x^2 + x + 1$
Message is $P(1) = 3, P(2) = 0, P(3) = 6$.

Error Correction: Berlekamp-Welch

Message: $m_1, \ldots, m_n$.
Sender:

1. Form degree $n-1$ polynomial $P(x)$ where $P(i) = m_i$.
2. Send $P(1), \ldots, P(n+2k)$.

Receiver:

1. Receive $R(1), \ldots, R(n+2k)$.
2. Solve $n+2k$ equations, $Q(i) = E(i)R(i)$ to find $Q(x) = E(x)P(x)$ and $E(x)$.
3. Compute $P(x) = Q(x)/E(x)$.
4. Compute $P(1), \ldots, P(n)$, recover the message.

Test Your Understanding

Say you sent a message of length 4, encoded as $P(x)$ where one sends packets $P(1), \ldots P(8)$.

You receive packets $R(1), \ldots R(8)$.

Packets 1 and 4 are corrupted.

Which options are True?

(A) $E(x) = (x-1)(x-4)$
(B) The number of coefficients in $E(x)$ is 2.
(C) The number of unknown coefficients in $E(x)$ is 2.
(D) $E(x) = (x-1)(x-2)$
(E) $R(4) \neq P(4)$
(F) The degree of $R(x)$ is 5.

Ans: (A), (C), (E).

A key question.

Is there one and only one $P(x)$ from Berlekamp-Welch procedure?

Existence: there is a $P(x)$ and $E(x)$ that satisfy equations.

Unique solution for $P(x)$?

Uniqueness: any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x). (holds \forall x) \quad (1)$$

Proof:
Suppose we assume the claim

$$Q'(x)E(x) = Q(x)E'(x) \text{ on } n+2k \text{ values of } x. \quad (2)$$

Proof that Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n+2k-1$
and agree on $n+2k$ points
$\implies Q'(x)E(x) = Q(x)E'(x)$.
Cross divide. □

Revisiting last bit.

Claim: $Q'(x)E(x) = Q(x)E'(x)$ on $n+2k$ values of x.

Proof: Construction implies that

$$Q(i) = R(i)E(i)$$
$$Q'(i) = R(i)E'(i)$$

for $i \in \{1, \ldots n+2k\}$.

If, for some i, $E(i) = 0$, then $Q(i) = 0$. If $E'(i) = 0$, then $Q'(i) = 0$.
$\implies Q(i)E'(i) = Q'(i)E(i)$ holds when $E(i)$ or $E'(i)$ are zero.

When $E'(i)$ and $E(i)$ are not zero

$$\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = R(i).$$

Cross multiplying gives equality in fact for these points. □

Points to polynomials, have to deal with zeros!

Berlekamp-Welch algorithm decodes correctly when at most k errors!

Summary. Error Correction.

Communicate n packets, with k erasures.

How many packets? $n+k$
How to encode? With polynomial, $P(x)$.
Of degree? $n-1$
Recover? Reconstruct $P(x)$ with any n points!

Communicate n packets, with k errors.

How many packets? $n+2k$
How to encode? With polynomial, $P(x)$. Of degree? $n-1$.
Recover?
Reconstruct error polynomial, $E(x)$, and $P(x)$!
Nonlinear equations.
Reconstruct $E(x)$ and $Q(x) = E(x)P(x)$. Linear Equations.
Polynomial division! $P(x) = Q(x)/E(x)$!

Reed-Solomon codes. Berlekamp-Welch Decoding. Efficient Solution!