CS 70 Discrete Mathematics and Probability Theory
Spring 2024 Seshia, Sinclair

DIS 5B

1 Countability: True or False

Note 11

(a) The set of all irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ (i.e. real numbers that are not rational) is uncountable.

(b) The set of integers x that solve the equation $3x \equiv 2 \pmod{10}$ is countably infinite.

(c) The set of real solutions for the equation $x + y = 1$ is countable.

For any two functions $f : Y \to Z$ and $g : X \to Y$, let their composition $f \circ g : X \to Z$ be given by $(f \circ g)(x) = f(g(x))$ for all $x \in X$. Determine if the following statements are true or false.

(d) f and g are injective (one-to-one) $\implies$ $f \circ g$ is injective (one-to-one).

(e) f is surjective (onto) $\implies$ $f \circ g$ is surjective (onto).

2 Counting Cartesian Products

Note 11

For two sets A and B, define the cartesian product as $A \times B = \{(a,b) : a \in A, b \in B\}$.

(a) Given two countable sets A and B, prove that $A \times B$ is countable.

(b) Given a finite number of countable sets $A_1, A_2, \ldots, A_n$, prove that

$$A_1 \times A_2 \times \cdots \times A_n$$

is countable.

(c) Consider a countably infinite number of finite sets: $B_1, B_2, \ldots$ for which each set has at least 2 elements. Prove that $B_1 \times B_2 \times \cdots$ is uncountable.

3 Hello World!

Note 12

Determine the computability of the following tasks. If it's not computable, write a reduction or self-reference proof. If it is, write the program.

(a) You want to determine whether a program P on input x prints "Hello World!". Is there a computer program that can perform this task? Justify your answer.

(b) You want to determine whether a program P prints "Hello World!" before running the kth line in the program. Is there a computer program that can perform this task? Justify your answer.

(c) You want to determine whether a program P prints "Hello World!" in the first k steps of its execution. Is there a computer program that can perform this task? Justify your answer.