# Outline for Today.

Polynomials.

Secret Sharing.

Finite Fields.

# Secret Sharing.

**Share secret among $n$ people.**

**Secrecy:** Any $k-1$ knows nothing.
**Robustness:** Any $k$ knows secret.

Geometric Intuition for today:

Two points make a line.
Lots of lines go through one point.

# Polynomials

A **polynomial**

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0.$$

is specified by **coefficients** $a_d, \ldots a_0$.

$P(x)$ **contains** point $(a, b)$ if $b = P(a)$.

**Polynomials over reals**: $a_1, \ldots, a_d \in \Re$, use $x \in \Re$.

# Field (in Mathematics)

Set with two commutative operations: addition and multiplication with additive/multiplicative identities and inverses
(except for additive identity has no multiplicative inverse).

E.g., Reals, rationals, complex numbers.
Not E.g., the integers.

Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.

# Finite Fields

Arithmetic modulo a prime integer $p$ has multiplicative inverses...

...and has only a finite number of elements.

Arithmetic modulo a prime $p$ is a **finite field** denoted by $GF(p)$.
$GF(p) = (\{0, \ldots, p-1\}, + \pmod{p}, * \pmod{p})$

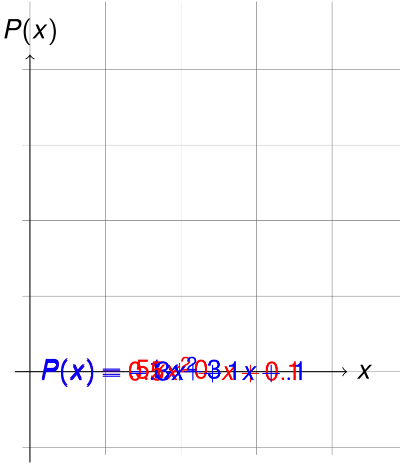
**Polynomials $P(x)$ with arithmetic modulo $p$:**

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0 \pmod{p},$$

for $x \in \{0, \ldots, p-1\}$ and $a_i \in \{0, \ldots, p-1\}$

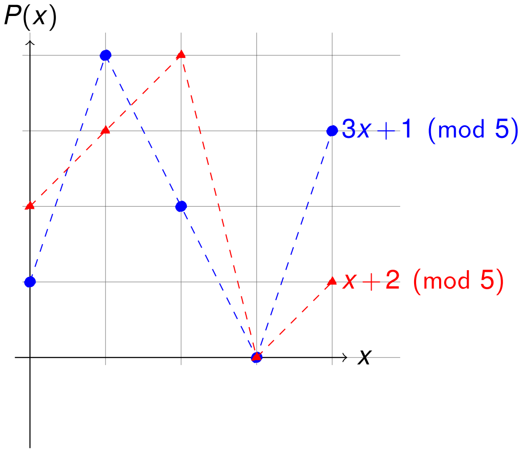
# Polynomial: $P(x) = a_d x^d + \cdots + a_0$ over $\Re$

Line: $P(x) = a_1 x + a_0 = mx + b$




Parabola: $P(x) = a_2 x^2 + a_1 x + a_0 = ax^2 + bx + c$

# Polynomial: $P(x) = a_d x^d + \cdots + a_0 \pmod p$




Finding an intersection.

$x + 2 \equiv 3x + 1 \pmod 5$

$\implies 2x \equiv 1 \pmod 5 \implies x \equiv 3 \pmod 5$

3 is multiplicative inverse of 2 modulo 5.

Good when modulus is prime!!

# Two points make a line.

**Fact:** Given $d+1$ points[1], exactly 1 degree $\leq d$ polynomial contains them.

Two points specify a line. Three points specify a parabola.

**Modular Arithmetic Fact:** Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime $p$ contains $d+1$ pts.

[1] Points with different $x$ values.

# Test your understanding: Polynominal Notation

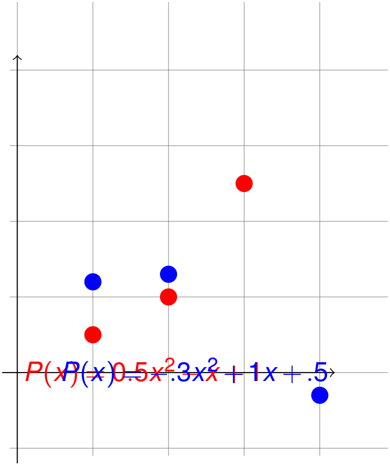
Polynomial: $a_n x^n + \cdots + a_0$.

**Consider line:** $mx + b$

(A) $a_1 = m$
(B) $a_1 = b$
(C) $a_0 = m$
(D) $a_0 = b$.

(A) and (D)

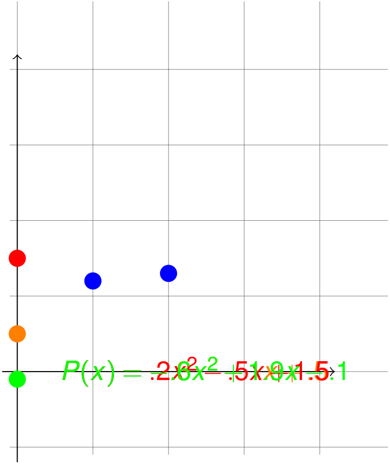
# 3 points determine a parabola.



**Fact:** Exactly 1 degree $\leq d$ polynomial contains $d + 1$ points. [2]

[2] Points with different $x$ values.

# 2 points not enough.



There is a $P(x)$ contains blue points and *any* $(0, y)$!

# Modular Arithmetic Fact and Secrets

**Modular Arithmetic Fact:** Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime $p$ contains $d+1$ pts.

**Shamir's $k$ out of $n$ Scheme:**
Secret $s \in \{0, \ldots, p-1\}$

1. Choose $a_0 = s$, and random $a_1, \ldots, a_{k-1}$.
2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_0$ with $a_0 = s$.
3. Share $i$ is point $(i, P(i) \mod p)$.

**Robustness:** Any $k$ shares gives secret.
Knowing $k$ pts $\implies$ only one $P(x)$ $\implies$ evaluate $P(0)$.
**Secrecy:** Any $k-1$ shares give nothing.
Knowing $\leq k-1$ pts $\implies$ any $P(0)$ is possible.

# Poll:example.

The polynomial from the scheme: $P(x) = 2x^2 + 1x + 3 \pmod 5$.
What is true for the secret sharing scheme using $P(x)$?

(A) The secret is "2".
(B) The secret is "3".
(C) A share could be $(1,5)$ because $P(1) = 5$
(D) A share could be $(2,4)$
(E) A share could be $(0,3)$

(B), (C) are true. (E) undesirable (reveals secret), start shares from $i = 1$.

# From $d+1$ points to degree $d$ polynomial?

For a line, $a_1 x + a_0 = mx + b$ contains points $(1,3)$ and $(2,4)$.

$$\begin{aligned} P(1) = m(1) + b &\equiv m + b \equiv 3 \pmod{5} \\ P(2) = m(2) + b &\equiv 2m + b \equiv 4 \pmod{5} \end{aligned}$$

Subtract first from second..

$$\begin{aligned} m + b &\equiv 3 \pmod{5} \\ m &\equiv 1 \pmod{5} \end{aligned}$$

Backsolve: $b \equiv 2 \pmod{5}$. Secret is 2.

And the line is...

$$x + 2 \mod 5.$$

# Quadratic

For a quadratic polynomial, $a_2x^2 + a_1x + a_0$ hits $(1,2);(2,4);(3,0)$.
Plug in points to find equations.

$$\begin{aligned} P(1) = a_2 + a_1 + a_0 &\equiv 2 \pmod 5 \\ P(2) = 4a_2 + 2a_1 + a_0 &\equiv 4 \pmod 5 \\ P(3) = 4a_2 + 3a_1 + a_0 &\equiv 0 \pmod 5 \end{aligned}$$

$$\begin{aligned} a_2 + a_1 + a_0 &\equiv 2 \pmod 5 \\ 3a_1 + 2a_0 &\equiv 1 \pmod 5 \\ 4a_1 + 2a_0 &\equiv 2 \pmod 5 \end{aligned}$$

Subtracting 2nd from 3rd yields: $a_1 = 1$.
$a_0 = (2 - 4(a_1))2^{-1} = (-2)(2^{-1}) = (3)(3) = 9 \equiv 4 \pmod 5$
$a_2 = 2 - 1 - 4 \equiv 2 \pmod 5$.

So polynomial is $2x^2 + 1x + 4 \pmod 5$

# In general..

Given points: $(x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)$.

Solve...

$$
\begin{aligned}
a_{k-1}x_1^{k-1} + \cdots + a_0 &\equiv y_1 \pmod{p} \\
a_{k-1}x_2^{k-1} + \cdots + a_0 &\equiv y_2 \pmod{p} \\
&\cdot \\
&\cdot \\
a_{k-1}x_k^{k-1} + \cdots + a_0 &\equiv y_k \pmod{p}
\end{aligned}
$$

Will this always work?

As long as solution **exists** and it is **unique!** And...

**Modular Arithmetic Fact:** Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime $p$ contains $d + 1$ pts.

# Another Construction: Interpolation!

For a quadratic, $a_2x^2 + a_1x + a_0$ hits $(1,2);(2,4);(3,0)$.

Find $\Delta_1(x)$ polynomial contains $(1,1);(2,0);(3,0)$.

Try $(x-2)(x-3) \pmod 5$.

Value is 0 at 2 and 3. Value is 2 at 1. Not 1!
So "Divide by 2" or multiply by 3.
$\Delta_1(x) = (x-2)(x-3)(3) \pmod 5$ contains $(1,1);(2,0);(3,0)$.

$\Delta_2(x) = (x-1)(x-3)(4) \pmod 5$ contains (1,0);(2,1);(3,0).

$\Delta_3(x) = (x-1)(x-2)(3) \pmod 5$ contains (1,0);(2,0);(3,1 ).

But wanted to hit $(1,2);(2,4);(3,0)$!

$P(x) = 2\Delta_1(x) + 4\Delta_2(x) + 0\Delta_3(x)$ works.

Same as before?

...after a lot of calculations... $P(x) = 2x^2 + 1x + 4 \mod 5$.

The same as before!

# Delta Polynomials: Concept.

For set of $x$-values, $x_1, \ldots, x_{d+1}$.

$$\Delta_i(x) = \begin{cases} 1, & \text{if } x = x_i. \\ 0, & \text{if } x = x_j \text{ for } j \neq i. \\ ?, & \text{otherwise.} \end{cases} \tag{1}$$

Given $d+1$ points, use $\Delta_i$ functions to go through points?
$(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$.

Will $y_1 \Delta_1(x)$ contain $(x_1, y_1)$?

Will $y_2 \Delta_2(x)$ contain $(x_2, y_2)$?

Does $y_1 \Delta_1(x) + y_2 \Delta_2(x)$ contain
$(x_1, y_1)$? and $(x_2, y_2)$?

See the idea? Function that contains all points?

$P(x) = y_1 \Delta_1(x) + y_2 \Delta_2(x) \ldots + y_{d+1} \Delta_{d+1}(x)$.

# There exists a polynomial...

**Modular Arithmetic Fact:** Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime $p$ contains $d+1$ pts.

**Proof of at least one polynomial:**
Given points: $(x_1, y_1); (x_2, y_2) \cdots (x_{d+1}, y_{d+1})$.

$$\Delta_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)} = \prod_{j \neq i}(x - x_j) \prod_{j \neq i}(x_i - x_j)^{-1}$$

Numerator is 0 at $x_j \neq x_i$.

"Denominator" makes it 1 at $x_i$.

And..

$$P(x) = y_1 \Delta_1(x) + y_2 \Delta_2(x) + \cdots + y_{d+1} \Delta_{d+1}(x).$$

hits points $(x_1, y_1); (x_2, y_2) \cdots (x_{d+1}, y_{d+1})$. Degree $d$ polynomial!

Construction proves the existence of a polynomial!

# Uniqueness.

**Uniqueness Fact.** At most one degree $d$ polynomial hits $d+1$ points.

**Roots fact:** Any nontrivial degree $d$ polynomial has at most $d$ roots.

Non-zero line (degree 1 polynomial) can intersect $y=0$ at only one $x$.

A parabola (degree 2), can intersect $y=0$ at only two $x$'s.

**Proof:**
Assume two different polynomials $Q(x)$ and $P(x)$ hit the points.

$R(x) = Q(x) - P(x)$ has $d+1$ roots and is degree $d$.
Contradiction.

□

Must prove **Roots fact.**

# Polynomial Division.

Divide $4x^2 - 3x + 2$ by $(x-3)$ modulo 5.

```
              4 x + 4 r 4
         -----------------
x - 3 )  4x^2 - 3 x + 2
         4x^2 - 2x
         ----------
                 4x + 2
                 4x - 2
                 -------
                      4
```

$4x^2 - 3x + 2 \equiv (x-3)(4x+4) + 4 \pmod{5}$

In general, divide $P(x)$ by $(x-a)$ gives $Q(x)$ and remainder $r$.

That is, $P(x) = (x-a)Q(x) + r$

# Only $d$ roots.

**Lemma 1:** $P(x)$ has root $a$ iff $P(x)/(x-a)$ has remainder 0:
$P(x) = (x-a)Q(x)$.

**Proof:** $P(x) = (x-a)Q(x) + r$.
Plugin $a$: $P(a) = r$.
It is a root if and only if $r = 0$.

□

**Lemma 2:** $P(x)$ has $d$ roots; $r_1, \ldots, r_d$ then
$P(x) = c(x-r_1)(x-r_2)\cdots(x-r_d)$.
**Proof Sketch:** By induction.

Induction Step: $P(x) = (x-r_1)Q(x)$ by Lemma 1.
$Q(x)$ has smaller degree so use the induction hypothesis. It has $d-1$ roots. Hence $Q(x) = c'(x-r_2)\cdots(x-r_d)$
Result follows. □

$d+1$ roots implies degree is at least $d+1$.

**Roots fact:** Any degree $d$ polynomial has at most $d$ roots.

# Secret Sharing: Summary

**Modular Arithmetic Fact:** Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d+1$ points.

**Shamir's $k$ out of $n$ Scheme:**
Secret $s \in \{0,\ldots,p-1\}$

1. Choose $a_0 = s$, and randomly $a_1,\ldots,a_{k-1}$.
2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_0$ with $a_0 = s$.
3. Share $i$ is point $(i, P(i) \mod p)$.

**Robustness:** Any $k$ knows secret.
Knowing $k$ pts, only one $P(x)$, evaluate $P(0)$.
**Secrecy:** Any $k-1$ knows nothing.
Knowing $\leq k-1$ pts, any $P(0)$ is possible.

# Minimality.

Need $p > n$ to hand out $n$ shares: $P(1) \ldots P(n)$.

For $b$-bit secret, must choose a prime $p > 2^b$.

**Theorem:** There is always a prime between $n$ and $2n$.

Working over numbers within 1 bit of secret size. **Minimality.**

With $k$ shares, reconstruct polynomial, $P(x)$.

With $k-1$ shares, any of $p$ values possible for $P(0)$!

# Runtime.

Runtime: polynomial in $k$, $n$, and $\log p$.

1. Evaluate degree $k - 1$ polynomial $n$ times using $\log p$-bit numbers.
2. Reconstruct secret by solving system of $k$ equations using $\log p$-bit arithmetic.

# A bit more counting.

What is the number of degree $d$ polynomials over $GF(m)$?

- $m^{d+1}$: $d+1$ coefficients from $\{0,\ldots,m-1\}$.
  coefficient representation
- $m^{d+1}$: $d+1$ points with $y$-values from $\{0,\ldots,m-1\}$
  value representation

Infinite number for reals, rationals, complex numbers!

# Summary

Two points make a line.

Compute solution: $m, b$.
Unique:
Assume two solutions, show they are the same.

Today: $d+1$ points make a unique degree $d$ polynomial.
Can solve linear system.
Solution exists: lagrange interpolation.
Unique:
Roots fact: Factoring: $(x-r)$ is root.
Induction only $d$ roots.
Apply: $P(x)$, $Q(x)$ degree $d$.
$P(x)-Q(x)$ is degree $d \implies d$ roots.
$P(x)=Q(x)$ on $d+1$ points $\implies P(x)=Q(x)$.

Secret Sharing:
$k$ points on degree $k-1$ polynomial is great!
Can hand out $n$ points on polynomial as shares.