Outline for Today.

Polynomials.

Secret Sharing.

Finite Fields.

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Share secret among *n* people.

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Geometric Intuition for today:

Two points make a line.
Lots of lines go through one point.

Polynomials

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is specified by **coefficients** $a_d, \ldots a_0$.

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Polynomials over reals: $a_1, \ldots, a_d \in \Re$, use $x \in \Re$.

Field (in Mathematics)

Set with two commutative operations: addition and multiplication with additive/multiplicative identities and inverses
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Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.

Finite Fields

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Polynomials $P(x)$ **with arithmetic modulo** p:

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0 \pmod p,$$

for $x \in \{0,\ldots,p-1\}$ and $a_i \in \{0,\ldots,p-1\}$

Polynomial: $P(x) = a_d x^d + \cdots + a_0$ over $\Re$

Line: $P(x) = a_1 x + a_0$

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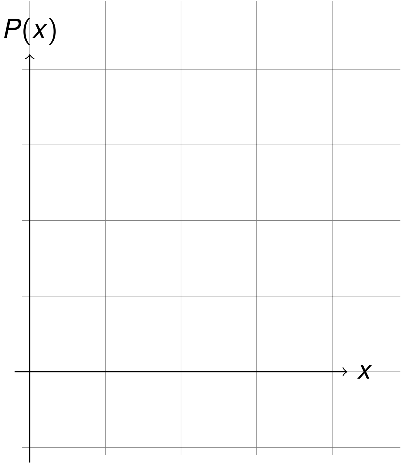

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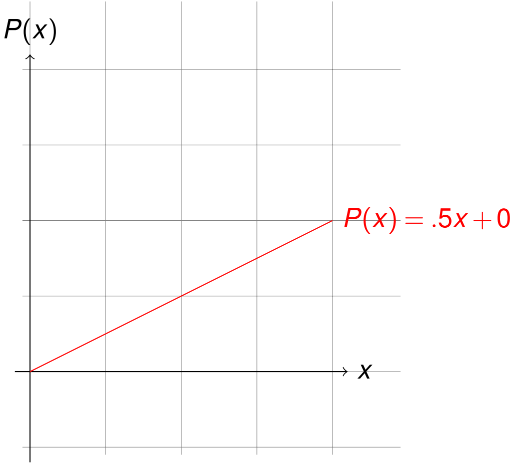

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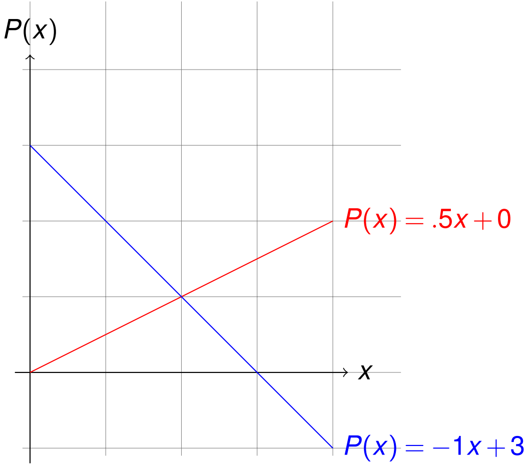

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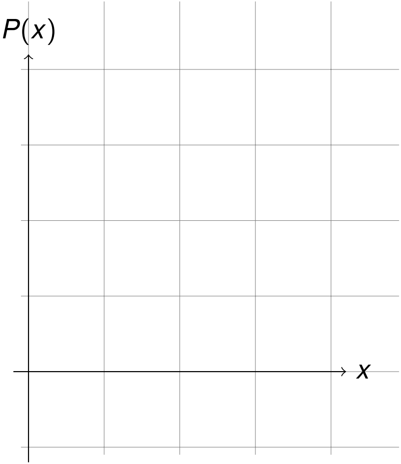

Parabola: $P(x) = a_2 x^2 + a_1 x + a_0$

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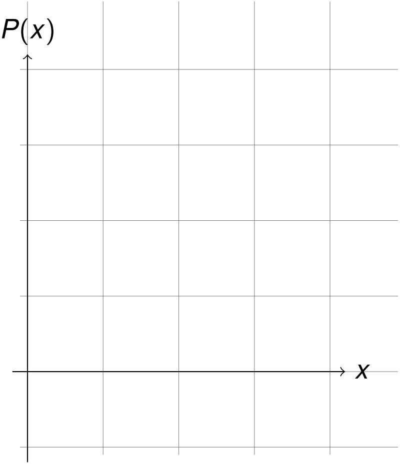

Parabola: $P(x) = a_2 x^2 + a_1 x + a_0 = ax^2 + bx + c$

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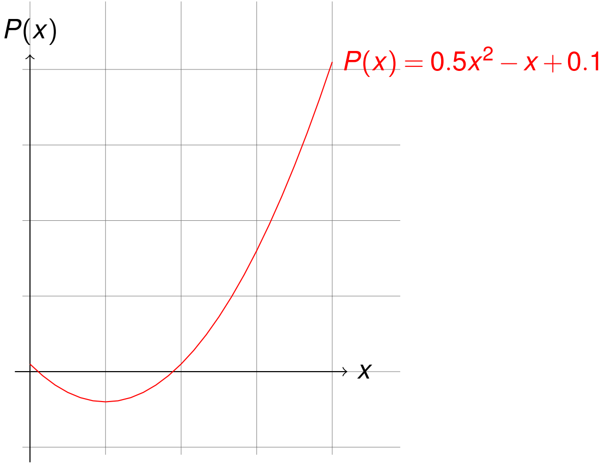

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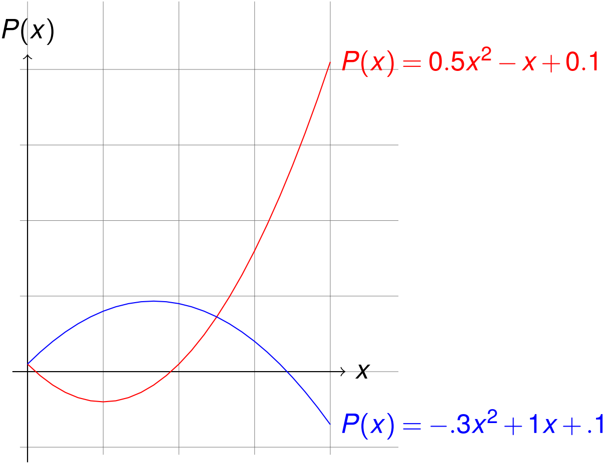

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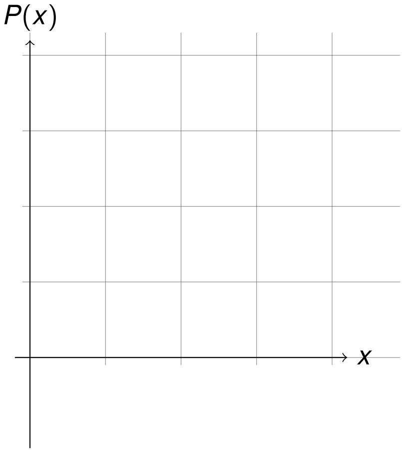

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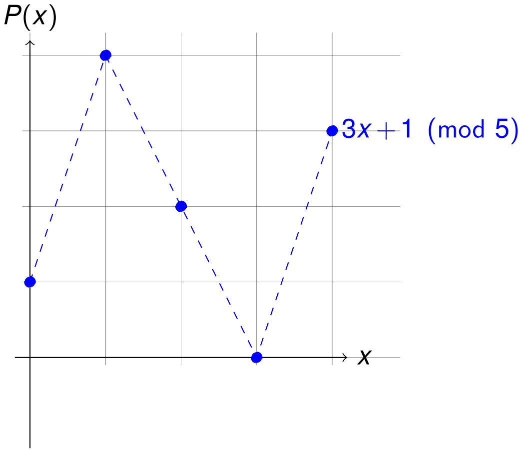

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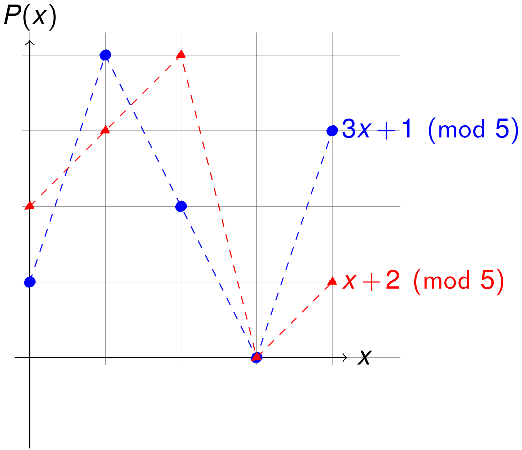

Finding an intersection.

$x+2 \equiv 3x+1 \pmod 5$

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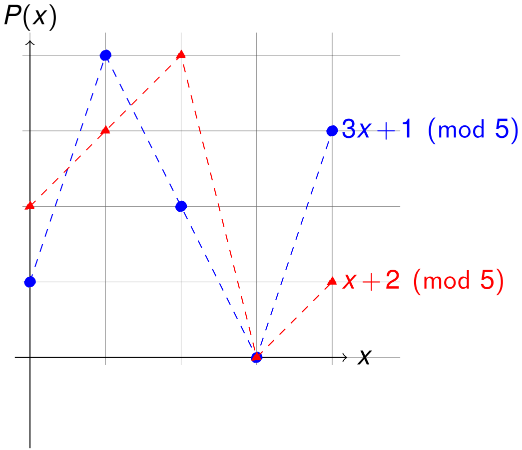

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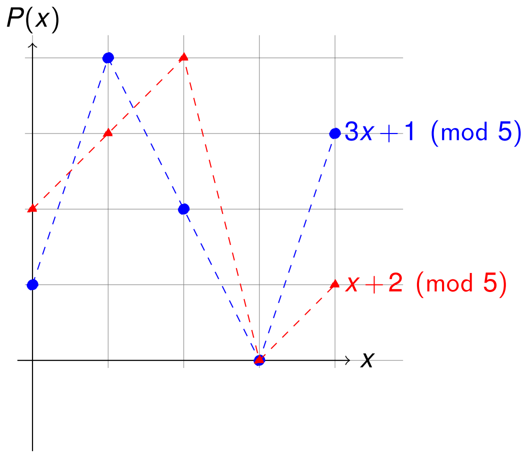

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3 is multiplicative inverse of 2 modulo 5.
Good when modulus is prime!!

Two points make a line.

Fact: Given $d+1$ points[1], exactly 1 degree $\leq d$ polynomial contains them.

[1] Points with different x values.

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Modular Arithmetic Fact: Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime p contains $d+1$ pts.

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Test your understanding: Polynominal Notation

Polynomial: $a_n x^n + \cdots + a_0$.

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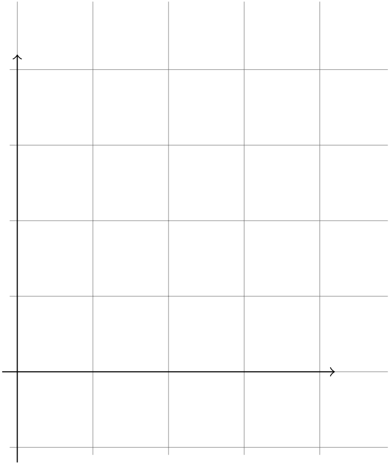
(A) and (D)

3 points determine a parabola.



Fact: Exactly 1 degree $\leq d$ polynomial contains $d + 1$ points. [2]

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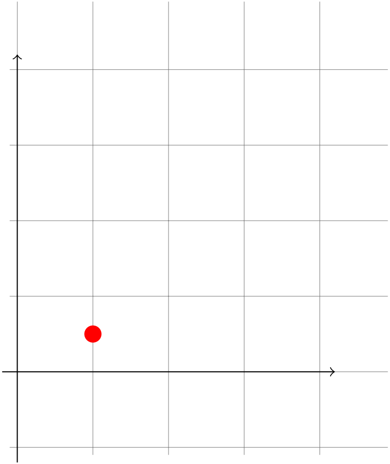

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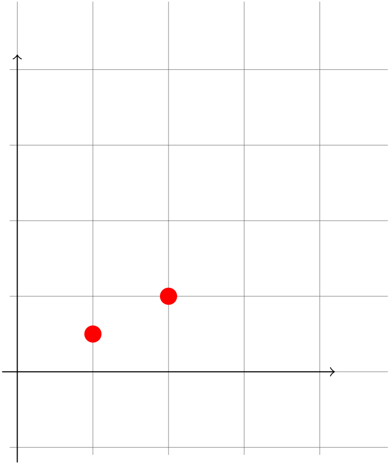

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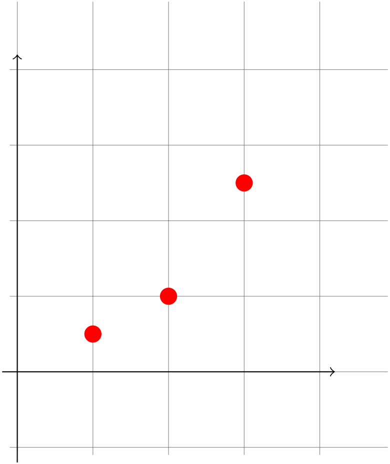

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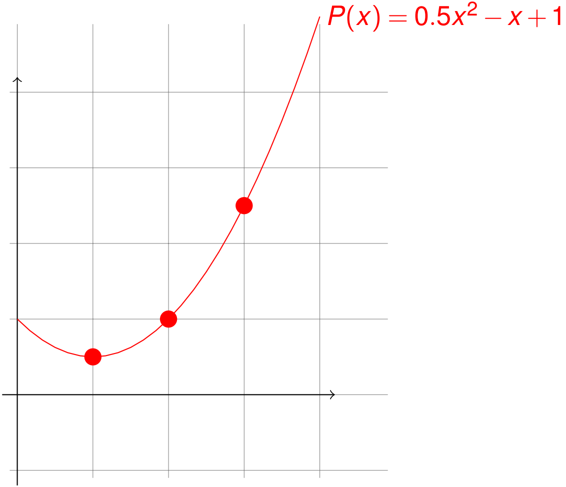

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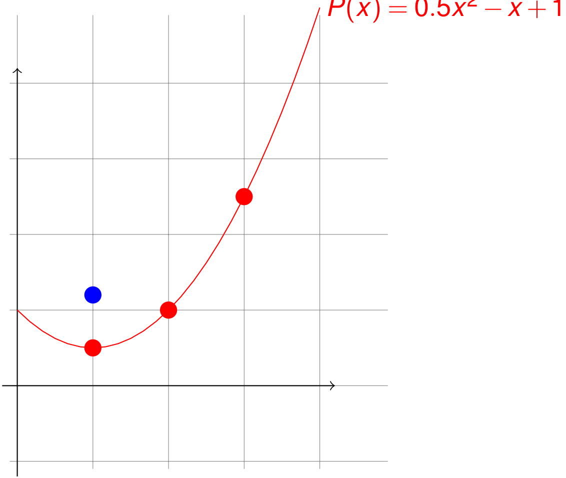

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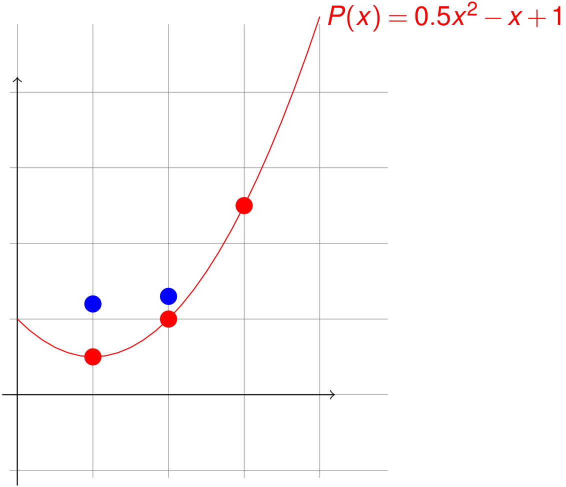

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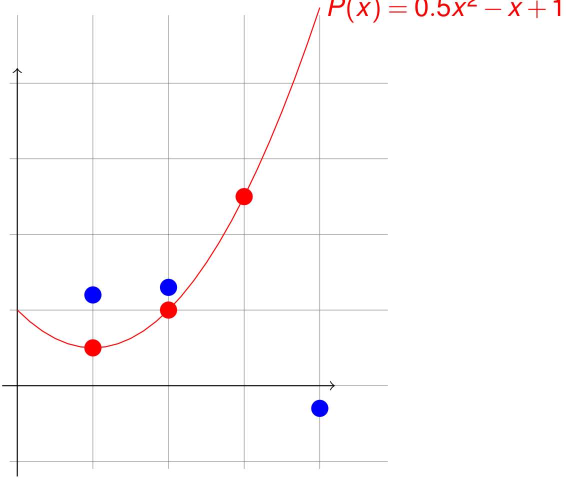

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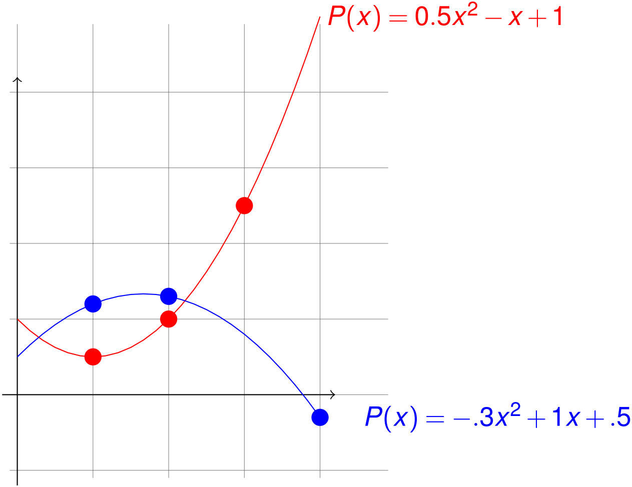

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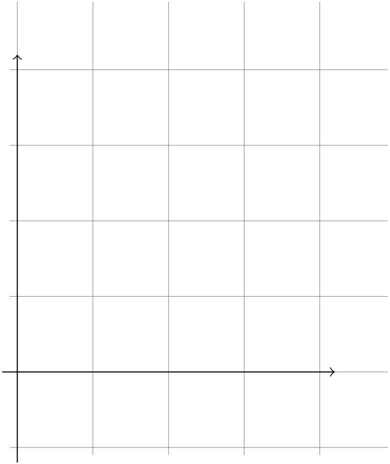
[2] Points with different x values.

2 points not enough.



There is a $P(x)$ contains blue points and *any* $(0, y)$!

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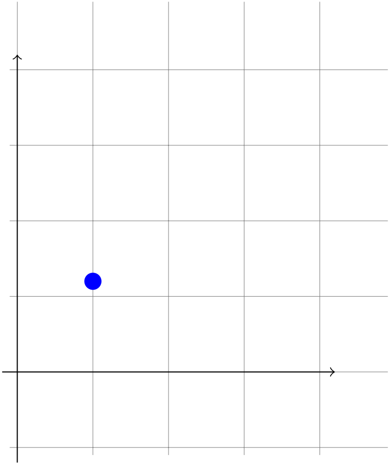

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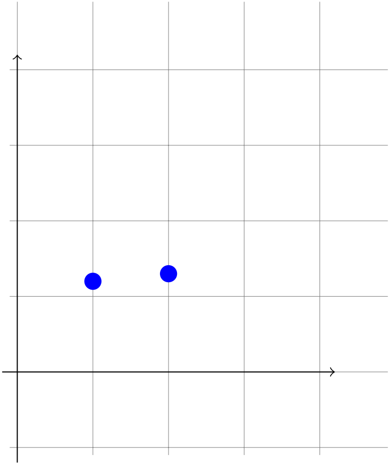

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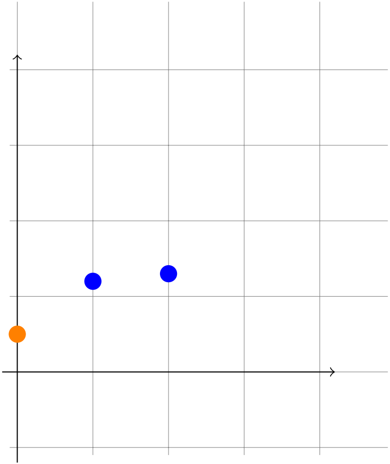

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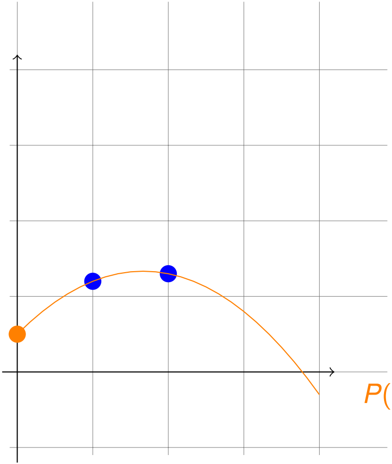

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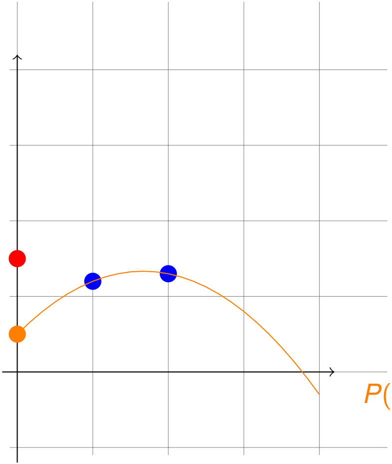

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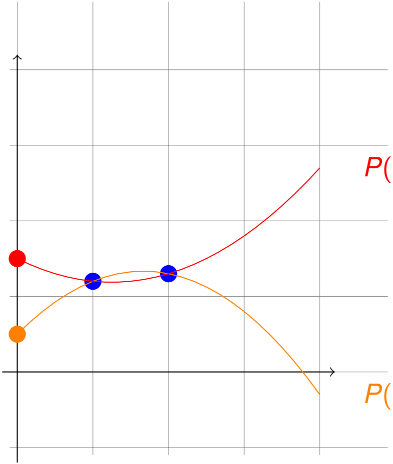

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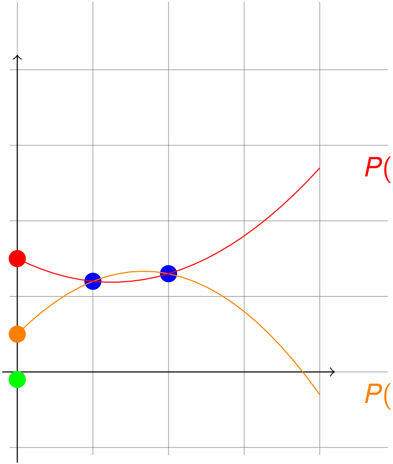

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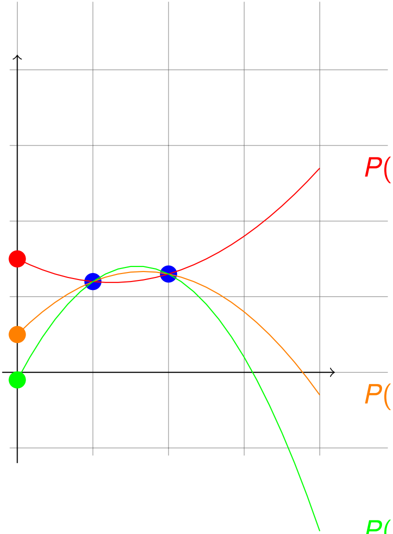

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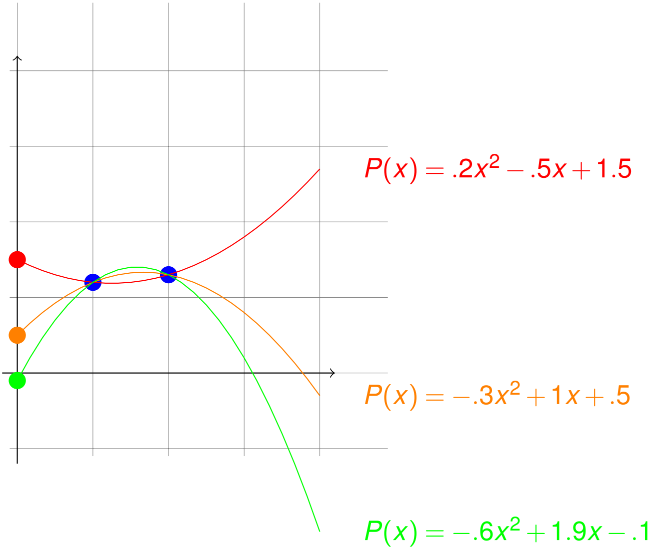

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Knowing k pts $\implies$ only one $P(x)$

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(B), (C) are true. (E) undesirable (reveals secret), start shares from $i = 1$.

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But wanted to hit $(1,2);(2,4);(3,0)$!

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Construction proves the existence of a polynomial!

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Divide $4x^2 - 3x + 2$ by $(x - 3)$ modulo 5.

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Only d roots.

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Roots fact: Any degree d polynomial has at most d roots.

Secret Sharing: Summary

Modular Arithmetic Fact: Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d+1$ points.

Shamir's k out of n Scheme:
Secret $s \in \{0,\ldots,p-1\}$

1. Choose $a_0 = s$, and randomly $a_1,\ldots,a_{k-1}$.
2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_0$ with $a_0 = s$.
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Robustness: Any k knows secret.
Knowing k pts, only one $P(x)$, evaluate $P(0)$.
Secrecy: Any $k-1$ knows nothing.
Knowing $\leq k-1$ pts, any $P(0)$ is possible.

Minimality.

Need $p > n$ to hand out n shares: $P(1) \ldots P(n)$.

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With $k-1$ shares, any of p values possible for $P(0)$!

Runtime.

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Runtime: polynomial in k, n, and $\log p$.

1. Evaluate degree $k-1$ polynomial n times using $\log p$-bit numbers.
2. Reconstruct secret by solving system of k equations using $\log p$-bit arithmetic.

A bit more counting.

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Infinite number for reals, rationals, complex numbers!

Summary

Two points make a line.

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Compute solution: m, b.
Unique:
Assume two solutions, show they are the same.

Today: $d+1$ points make a unique degree d polynomial.
Can solve linear system.
Solution exists: lagrange interpolation.
Unique:
Roots fact: Factoring: $(x-r)$ is root.
Induction only d roots.
Apply: $P(x)$, $Q(x)$ degree d.
$P(x)-Q(x)$ is degree $d \implies d$ roots.
$P(x) = Q(x)$ on $d+1$ points $\implies P(x) = Q(x)$.

Secret Sharing:
k points on degree $k-1$ polynomial is great!
Can hand out n points on polynomial as shares.