Outline

- Erasure Codes
- Error Correction

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- Error Correction
- More Polynomials!

Erasure Codes.

Satellite

GPS device

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3 packet message.

GPS device

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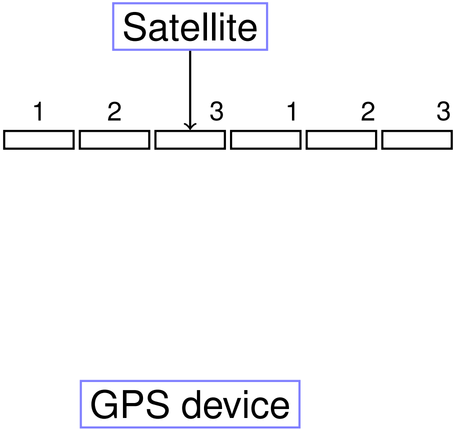
Satellite

3 packet message.

Lose 3 out 6 packets.

GPS device

Erasure Codes.



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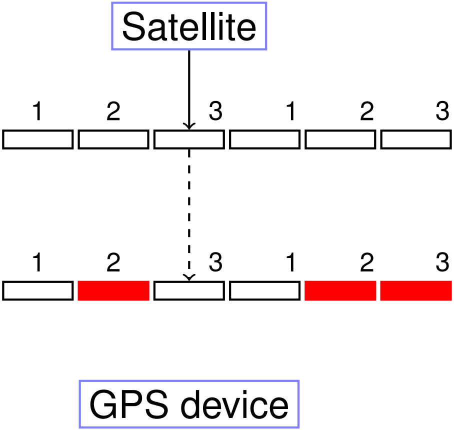

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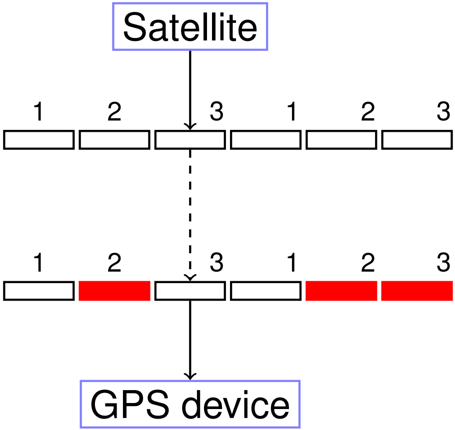

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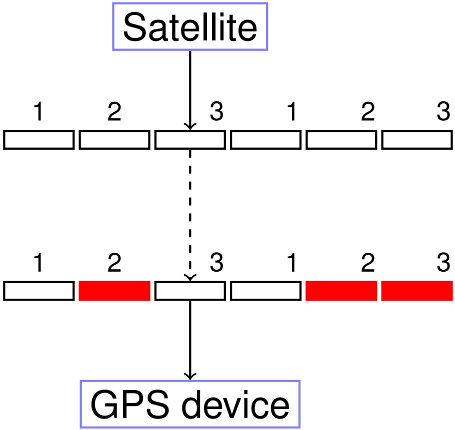

3 packet message. So send 6!

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Gets packets 1,1,and 3.

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Solution Idea: Use Polynomials!!!

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n packet message, channel that loses k packets.

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n packet message.

GPS device

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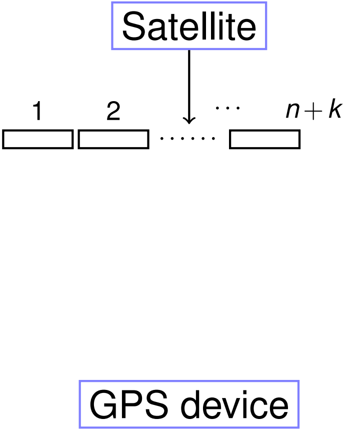
Lose *k* packets.

GPS device

Erasure Codes.



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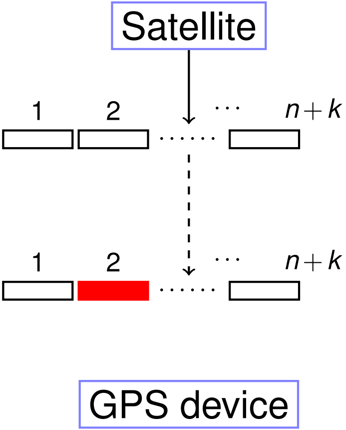

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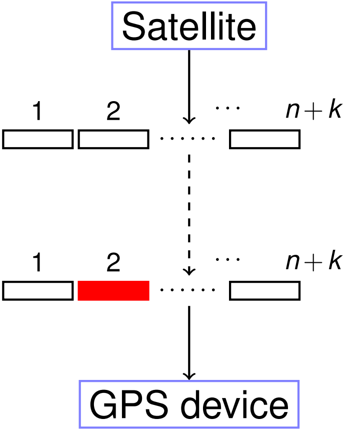

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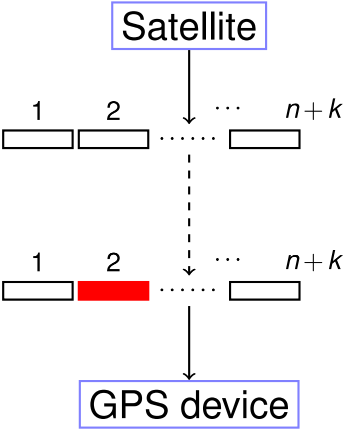

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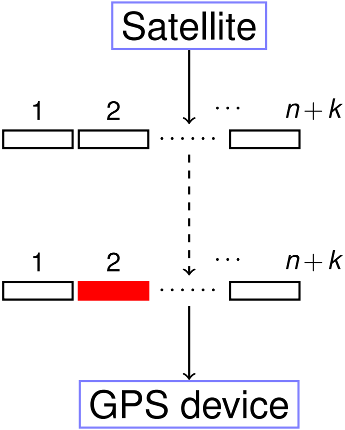

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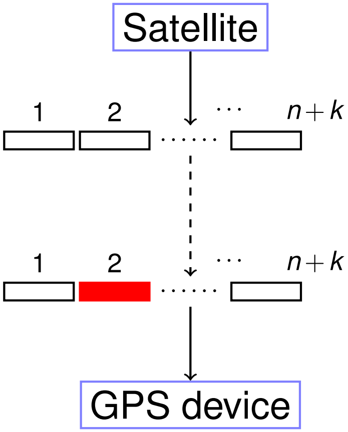

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n packet message.

Optimal.

Comparison with Secret Sharing.

Comparing information content:

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Secret Sharing: each share is size of whole secret.

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Coding: Each packet has size $1/n$ of the whole message.

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Send message of 1,4, and 4.

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Send message of 1,4, and 4. up to 3 erasures.

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Send message of 1,4, and 4. up to 3 erasures. $n = 3, k = 3$

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Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$.

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$P(x) = x^2 \pmod{5}$

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6 points. Better work modulo 7 at least!

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$$P(1) = a_2 + a_1 + a_0 \equiv 1 \pmod{7}$$

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$a_1 = 2a_0$. $a_0 = 2 \pmod{7}$ $a_1 = 4 \pmod{7}$ $a_2 = 2 \pmod{7}$

$P(x) = 2x^2 + 4x + 2$

Example

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Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$.

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Notice that packets are of the form x, y: contain "x-values".

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- Give Erasure Codes:
  Send $n+k$ pairs (x,y) to reconstruct n-packet message

Next: Error Correction

Noisy Channel: corrupts k packets. (rather than loss/erasures.)

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Additional Challenge: Finding which packets are corrupt.

Error Correction

Satellite

GPS device

Error Correction

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3 packet message.

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3 packet message.

Corrupts 1 packets.

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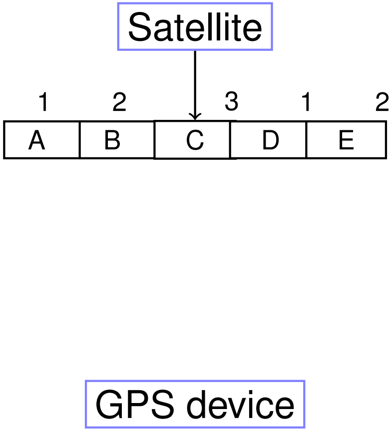
Error Correction



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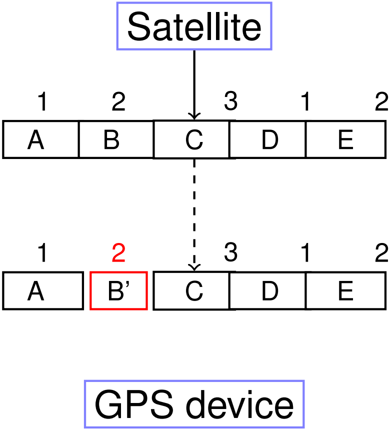

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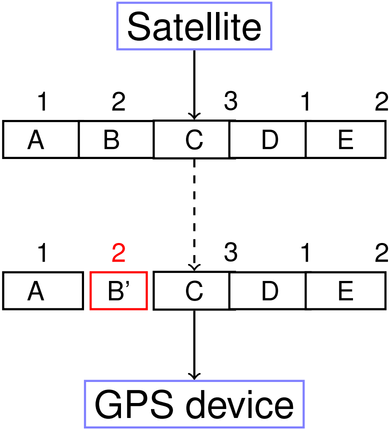

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that contains $\geq n+k$ received points.

Proof:

(1) Easy. Only k corruptions (by assumption).
(2) Degree $n-1$ polynomial $Q(x)$ consistent with $n+k$ points.
$Q(x)$ agrees with $R(i)$, $n+k$ times.
$P(x)$ agrees with $R(i)$, $n+k$ times. (possibly different $n+k$ from above?)

Total points contained by both	: $2n+2k$.	P	Pigeons.
Total points to choose from	: $n+2k$.	H	Holes.
Points contained by both	: $\geq n$.	$\geq P-H$	Collisions.

$\implies Q(i) = P(i)$ at n points.
$\implies Q(x) = P(x)$. □

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$P(i) = R(i)$ for $n + k = 3 + 1 = 4$ points.

Slow solution.

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Reconstructs $P(x)$ and only $P(x)$!!

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How do we find where the bad packets are efficiently?!?!?!

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4 unknowns (p_0, p_1, p_2 and e), 5 nonlinear equations.

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Note: this is linear in a_i and coefficients of $E(x)$!

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Solving for $Q(x)$ and $E(x)$...

For all points $1, \ldots, i, n+2k$,

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..and $n+2k$ unknown coefficients of $Q(x)$ and $E(x)$!

Solving for $Q(x)$ and $E(x)$...

For all points $1, \dots, i, n+2k$,

$$Q(i) = R(i)E(i) \pmod{p}$$

Gives $n+2k$ linear equations.

$$\begin{aligned}
a_{n+k-1} + \dots a_0 &\equiv R(1)(1 + b_{k-1} \cdots b_0) \pmod{p} \\
a_{n+k-1}(2)^{n+k-1} + \dots a_0 &\equiv R(2)((2)^k + b_{k-1}(2)^{k-1} \cdots b_0) \pmod{p} \\
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a_{n+k-1}(m)^{n+k-1} + \dots a_0 &\equiv R(m)((m)^k + b_{k-1}(m)^{k-1} \cdots b_0) \pmod{p}
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          ------------------
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```

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```
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              -----------------
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         ----------
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```
                  1 x^2 + 1 x
             -----------------
x - 2 )  x^3  + 6 x^2 + 6 x + 5
         x^3  - 2 x^2
         ----------
                  1 x^2 + 6 x + 5
                  1 x^2 - 2 x
                  --------------
                            x + 5
```

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$Q(x) = x^3 + 6x^2 + 6x + 5.$
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```
                 1 x^2 + 1 x + 1
             -----------------
x - 2 )  x^3  + 6 x^2 + 6 x + 5
         x^3  - 2 x^2
         ----------
                 1 x^2 + 6 x + 5
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                 ---------------
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                           -----
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                             -----
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                          -----
                              0
```

$P(x) = x^2 + x + 1$
Message is $P(1) = 3, P(2) = 0, P(3) = 6$.

Error Correction: Berlekamp-Welch

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3. Compute $P(x) = Q(x)/E(x)$.
4. Compute $P(1), \ldots, P(n)$, recover the message.

Test Your Understanding

Say you sent a message of length 4, encoded as $P(x)$ where one sends packets $P(1), ...P(8)$.

You receive packets $R(1), ...R(8)$.

Packets 1 and 4 are corrupted.

Which options are True?

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Ans: (A), (C), (E).

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Is there one and only one $P(x)$ from Berlekamp-Welch procedure?

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Is there one and only one $P(x)$ from Berlekamp-Welch procedure?

Existence: there is a $P(x)$ and $E(x)$ that satisfy equations.

Unique solution for $P(x)$?

Uniqueness: any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).(holds \forall x) \tag{1}$$

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for $i \in \{1, \ldots n+2k\}$.

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If, for some i, $E(i) = 0$, then $Q(i) = 0$.

Revisiting last bit.

Claim: $Q'(x)E(x) = Q(x)E'(x)$ on $n+2k$ values of x.

Proof: Construction implies that

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Points to polynomials, have to deal with zeros!

Berlekamp-Welch algorithm decodes correctly when at most k errors!

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Communicate n packets, with k erasures.

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How many packets?

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How many packets? $n+k$
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Reed-Solomon codes.

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Reed-Solomon codes. Berlekamp-Welch Decoding. Efficient Solution!