How big is the set of reals or the set of integers?

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Infinite!

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Infinite!

Is one bigger or smaller?

Same Size?

When are two sets the same size?
(A) Bijection between the sets.
(B) Count the objects in each and get the same number.
(C) Both sets are infinite.

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(A), (B).
Not (C)... at least, not always! We will see why.

Countable.

How to count?

Countable.

How to count?

0,

Countable.

How to count?

0, 1,

Countable.

How to count?

0, 1, 2,

Countable.

How to count?

0, 1, 2, 3,

Countable.

How to count?

$0, 1, 2, 3, \ldots$

Countable.

How to count?

0, 1, 2, 3, $\ldots$

The Counting numbers.

Countable.

How to count?

0, 1, 2, 3, $\ldots$

The Counting numbers.
The natural numbers! N

Countable.

How to count?

0, 1, 2, 3, . . .

The Counting numbers.
The natural numbers! N

Definition: S is **countable** if there is a bijection between S and some subset of N.

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.
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Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

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Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

If the subset of N is infinite, S is **countably infinite**.

Z^+ vs. N: Where's 0?

Which is bigger?

The positive integers, Z^+, or the natural numbers, N.

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More natural numbers!?

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Consider $f : Z^+ \to N$ where $f(z) = z - 1$.

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But.. where's zero? "It comes from 1."

More sets.

E - Even natural numbers. Countable?

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Evens are countably infinite.

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Evens are countably infinite.
Evens are same size as all natural numbers.

All integers?

What about Integers, Z?

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Define $f : N \rightarrow Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

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One-to-one: For $x \neq y$
if x is even and y is odd,
then $f(x)$ is nonnegative and $f(y)$ is negative

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Onto: For any $z \in Z$,
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Integers and naturals have same size!

Listings..

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n	$f(n)$

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0	0

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Another View:

n	$f(n)$
0	0
1	-1

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n	$f(n)$
0	0
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2	1

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n	$f(n)$
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1	-1
2	1
3	-2

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Notice that: A listing “is” a bijection with a subset of natural numbers.

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Notice that: A listing "is" a bijection with a subset of natural numbers.
If finite: bijection with $\{0,\dots,|S|-1\}$
If infinite: bijection with N.

Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

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"Output element of S",
"Output next element of S"

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Any element x of S has *specific, finite* position in list.

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$\ldots$

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Consider the integers Z:

$Z = \{0, 1,$

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"Output element of *S*",
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Enumerability $\equiv$ countability.

Enumerating (listing) a set implies that it is countable.

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Need to be careful.

Countably infinite subsets.

Enumerating a set implies countable.

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Implications:
Z^+ is countable (because Z is countable).

Enumeration example.

All binary strings.

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Never get to 1!

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Can't list in "order".

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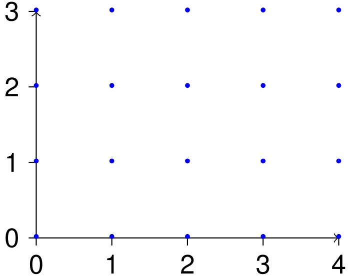

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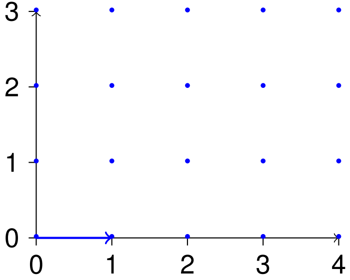

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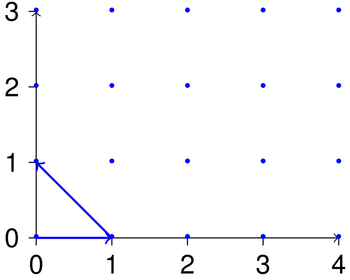

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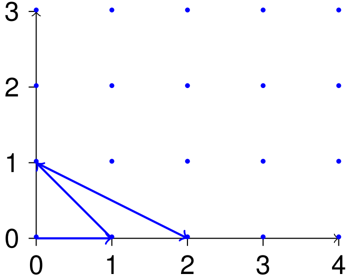

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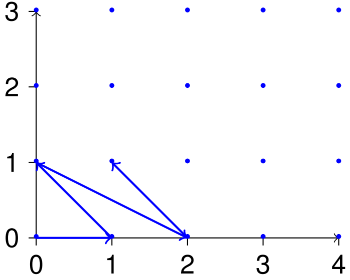

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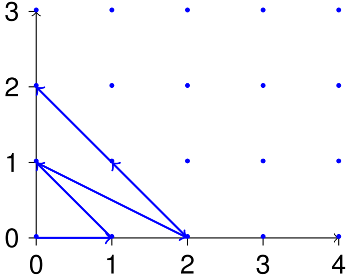

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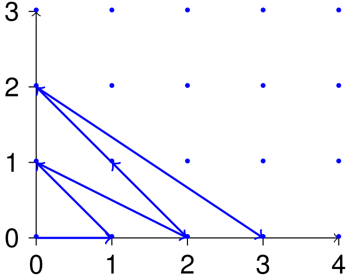

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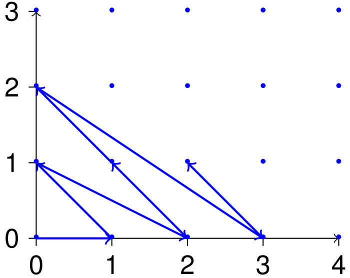

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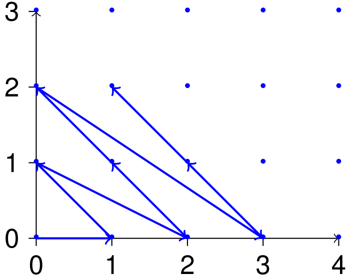

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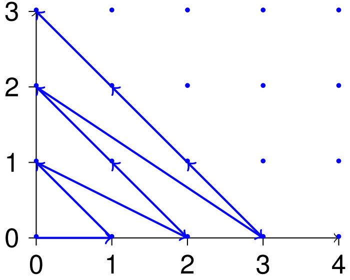

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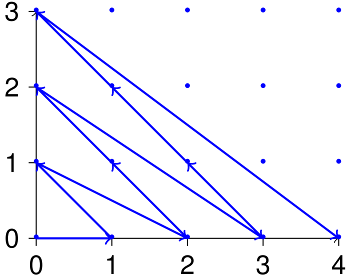

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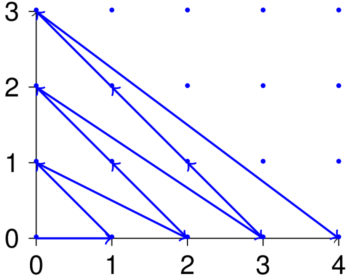

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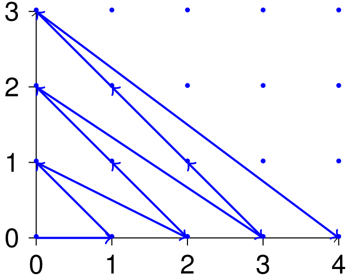

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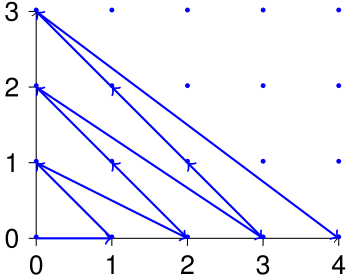

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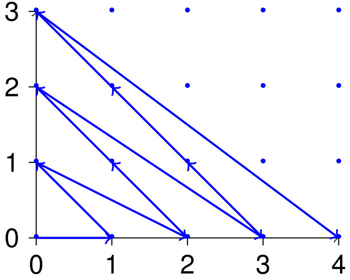

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We will use this representation to answer the question above!

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$\vdots$

Construct "diagonal" number:

Diagonalization.

Assume countable. There is a listing, L contains all reals. For example

0: .500000000...
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⋮

Construct "diagonal" number: .7

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Construct "diagonal" number: .77

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Subset $[0,1]$ is not countable!!

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If reals are countable then so is $[0,1]$.

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$[0,1]$ is same cardinality as nonnegative reals!

Another diagonalization.

The set of all subsets of N.

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Example subsets of N: $\{0\}$,

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Assume is countable.

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If *i*th set in *L* does not contain *i*, $i \in D$.

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Theorem: The set of all subsets of N is not countable.

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Theorem: The set of all subsets of N is not countable.
(The set of all subsets of S, is the **powerset** of N.)

Poll: Which of these are true?

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(A) Integers are larger than Naturals.
(B) Integers are countable.
(C) Reals can't be enumerated: diagonal number not on list.
(D) Powerset of Naturals can be enumerated.

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(B), (C)

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