Barber paradox.

Created by logician Bertrand Russell.

Village with just 1 barber (a man), all men clean-shaven.
Barber announces:
"I shave all and only those men who do not shave themselves."

Who shaves the barber?

Case 1: It's the barber.
Case 2: Somebody else.

Cannot answer that question in either case! Paradox!!!

Russell's Paradox: Assuming Existence of Set of All Sets

Naive Set Theory: Any definable collection is a set.

$$\exists y\, \forall x\, (x \in y \iff P(x)) \tag{1}$$

y is the set of elements that satisfies the proposition $P(x)$.

$P(x) = x \notin x$.

There exists a y that satisfies statement 1 for $P(\cdot)$.

Take $x = y$.

$$y \in y \iff y \notin y.$$

Contradiction!

Is this stuff actually useful?

Problem 1: Verify that my program is correct!

Problem 2: Check that the compiler works correctly!
(output program is equivalent to its input program)

How about.. Check that the compiler terminates on a certain input.

$HALT(P, I)$
P - program
I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Notice:
Need a computer
...with the notion of a stored program!!!!
(not an adding machine! not a person and an adding machine.)

Program is a text string.
Text string can be an input to a program.
Program can be an input to a program.

Implementing HALT.

$HALT(P, I)$
P - program
I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Run P on I and check!

How long do you wait?

Halt does not exist.

$HALT(P, I)$
P - program
I - input.

Determines if $P(I)$ (P run on I) halts or loops forever.

Theorem: There is no program HALT.

Proof Idea: Proof by contradiction, use self-reference.

Halt and Turing.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Turing(P)
1. If HALT(P,P) ="halts", then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program HALT.
There is text that "is" the program HALT.
There is text that is the program Turing.
Can run Turing on Turing!

Does Turing(Turing) halt?

Case 1: Turing(Turing) halts
$\implies$ then HALT(Turing, Turing) $=$ halts
$\implies$ Turing(Turing) loops forever.

Case 2: Turing(Turing) loops forever
$\implies$ then HALT(Turing, Turing) $\neq$ halts
$\implies$ Turing(Turing) halts.

Contradiction. Program HALT does not exist! $\square$

Another view of proof: diagonalization.

Any program is a fixed length string.
Fixed length strings are enumerable.
Program halts or not any input, which is a string.

	P_1	P_2	P_3	$\cdots$
P_1	H	H	L	$\cdots$
P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Halt(P,P) - diagonal.
Turing - is not Halt.
and is different from every P_i on the diagonal.
Turing is not on list. $\implies$ Turing is not a program.
But Turing can be constructed as a program if the program Halt exists.
Halt does not exist! □

Summary: computability.

Computer Programs are interesting objects.
Mathematical objects.
Formal Systems.

Computer Programs cannot completely "understand" computer programs.

Example: no computer program can tell if any other computer program HALTS.

Proof Idea: Diagonalization.
Program: Turing (or DIAGONAL) takes P.
Assume there is HALT.
DIAGONAL flips answer.
Loops if P halts, halts if P loops.
What does Turing do on turing? Doesn't loop or HALT.
HALT does not exist! □

More on this topic in CS 172.

Computation is a lens for other action in the world.

Turing machine.

A Turing machine.
– an (infinite) tape with characters
– be in a state, and read a character
– move left, right, and/or write a character.

Universal Turing machine
– an interpreter program for a Turing machine
– where the tape could be a description of a ... Turing machine!

Now that's a computer! (not far from today's computers)

Church, Gödel and Turing.

Church proved an equivalent theorem. (Previously.)

Used λ calculus....which is... a programming language!!!
Just like Python, C, Javascript, ...

Gödel: Incompleteness theorem.

Any formal system either is inconsistent or incomplete.
Inconsistent: A false sentence can be proven.
Incomplete: There is no proof for some sentence in the system.

Along the way: "built" computers out of arithmetic.
Showed that every mathematical statement corresponds to an ....natural number!!!!