## Counting and Probability

Second half of the semester: Probability.

A bag contains a set of colored balls:

What is the chance that a ball taken from the bag is blue?

Count blue. Count total. Divide.

Today: Counting!

After the Midterm: Probability. Professor Sinclair.

## Outline

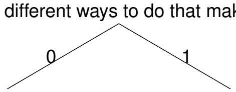
1. Counting.
2. Tree
3. Rules of Counting
4. Sample with/without replacement where order does/doesn't matter.

## Count?

How many outcomes possible for $k$ coin tosses?
How many handshakes for $n$ people?
How many 10 digit numbers?
How many 10 digit numbers without repeating digits?

## Using a tree of possibilities...

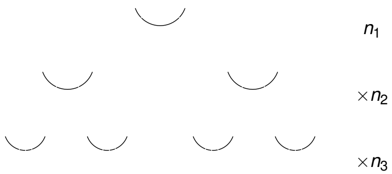
How many 3-bit strings?
How many different sequences of three bits from $\{0, 1\}$?
How would you make one sequence?
How many different ways to do that making?




8 leaves which is $2 \times 2 \times 2$. One leaf for each string.
8 3-bit strings!

## First Rule of Counting: Product Rule

Objects made by choosing from $n_1$, then $n_2$, ..., then $n_k$
the number of objects is $n_1 \times n_2 \cdots \times n_k$.




In picture, $2 \times 2 \times 3 = 12$

## Using the first rule..

How many outcomes possible for $k$ coin tosses?

2 ways for first choice, 2 ways for second choice, ...
$2 \times 2 \cdots \times 2 = 2^k$

How many 10 digit numbers (leading zeroes are OK)?

10 ways for first choice, 10 ways for second choice, ...
$10 \times 10 \cdots \times 10 = 10^{10}$

How many 10 digit numbers (no leading zeroes)?

9 ways for first choice, 10 ways for second choice, ...
$9 \times 10 \cdots \times 10 = 9 \times 10^9$

How many $n$ digit base $m$ numbers?

$m$ ways for first, $m$ ways for second, ...
$m^n$

## Functions, polynomials.

How many functions $f$ mapping $S$ to $T$?

$|T|$ ways to choose for $f(s_1)$, $|T|$ ways to choose for $f(s_2)$, ...

....$|T|^{|S|}$

How many polynomials of degree at most $d$ modulo $p$?

$p$ ways to choose for first coefficient, $p$ ways for second, ...
...$p^{d+1}$

$p$ values for first point, $p$ values for second, ...
...$p^{d+1}$

## Permutations.

How many 10 digit numbers **without repeating a digit**?

10 ways for first, 9 ways for second, 8 ways for third, ...

... $10 * 9 * 8 \cdots * 1 = 10!$.[1]

How many different samples of size $k$ from $n$ numbers **without replacement.**

$n$ ways for first choice, $n-1$ ways for second,
$n-2$ choices for third, ...

... $n * (n-1) * (n-2) \cdot * (n-k+1) = \frac{n!}{(n-k)!}$.

How many orderings of $n$ objects are there?
**Permutations of $n$ objects.**

$n$ ways for first, $n-1$ ways for second,
$n-2$ ways for third, ...

... $n * (n-1) * (n-2) \cdot * 1 = n!$.

[1] By definition: $0! = 1$. $n! = n(n-1)(n-2)\ldots 1$.

## One-to-One Functions.

How many one-to-one functions from $S$ to $S$?

$|S|$ choices for $f(s_1)$, $|S|-1$ choices for $f(s_2)$, ...

So total number is $|S| \times |S|-1 \cdots 1 = |S|!$

A one-to-one function (from $S$ to $S$) is a permutation!

## Counting sets..when order doesn't matter.

How many sets of 5 playing cards ("poker hands")?

$52 \times 51 \times 50 \times 49 \times 48$ ???

Are $A, K, Q, 10, J$ of spades
and $10, J, Q, K, A$ of spades the same?
**Second Rule of Counting:** If order doesn't matter count ordered objects and then divide by number of orderings.[2]

Number of orderings for a poker hand: 5!

Can write as...

$$\frac{52 \times 51 \times 50 \times 49 \times 48}{5!}$$

$$\frac{52!}{5! \times 47!}$$

Generic: ways to choose 5 out of 52 possibilities.

[2] When each unordered object corresponds equal numbers of ordered objects.

## When order doesn't matter.

Choose 2 out of $n$?

$$\frac{n \times (n-1)}{2} = \frac{n!}{(n-2)! \times 2}$$

Choose 3 out of $n$?

$$\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}$$

Choose $k$ **out of** $n$?

$$\frac{n!}{(n-k)! \times k!}$$

**Notation: $\binom{n}{k}$ and pronounced "$n$ choose $k$."**

## Simple Practice.

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways to choose second, 1 for last.
$\implies 3 \times 2 \times 1 = 3!$ orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A!
total orderings of 7 letters. 7!
total "extra counts" or orderings of two A's? 3!
Total orderings? $\frac{7!}{3!}$

How many orderings of letters in MISSISSIPPI?

4 S's, 4 I's, 2 P's.
11 letters total!
11! ordered objects!
$4! \times 4! \times 2!$ ordered objects per "unordered object"
$\implies \frac{11!}{4!4!2!}$.

## Sampling...

Sample $k$ items out of $n$

Without replacement:
Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$
Order does not matter:
Second Rule: divide by number of orders – "$k!$"
$\implies \frac{n!}{(n-k)!k!}$.
"$n$ choose $k$"

With Replacement.
Order matters: $n \times n \times \ldots n = n^k$
Order does not matter: Second rule ???

Problem: depends on how many of each item we chose!

Set: $1,2,3$ $3!$ orderings map to it.
Set: $1,2,2$ $\frac{3!}{2!}$ orderings map to it.

How do we deal with this situation?!?!

## New Technique: Stars and Bars....

How many ways can Bob and Alice split 5 dollars?
For each of 5 dollars pick Bob or Alice($2^5$), see what results.

5 dollars for Bob and 0 for Alice:
one ordered set: $(B,B,B,B,B)$.

4 for Bob and 1 for Alice:
5 ordered sets: $(A,B,B,B,B)$ ; $(B,A,B,B,B)$; ...

Well, we can list the possibilities.
$0+5$, $1+4$, $2+3$, $3+2$, $4+1$, $5+0$.

For 2 numbers adding to $k$, we get $k+1$.

For 3 numbers adding to $k$? More than 3?

## Stars and Bars.

How many ways to add up $n$ natural numbers to equal $k$?

Or: $k$ choices from set of $n$ possibilities with replacement.
**Sample with replacement where order just doesn't matter.**

How many ways can Alice, Bob, and Eve split 5 dollars.

Think of Five dollars as Five stars: $\star\star\star\star\star$.

Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: $\star\star|\star|\star\star$.

Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: $|\star|\star\star\star\star$.

Each split $\implies$ a sequence of stars and bars.
Each sequence of stars and bars $\implies$ a split.

**Counting Rule: if there is a one-to-one mapping between two sets they have the same size!**

## Stars and Bars.

How many different 5 star and 2 bar diagrams?

7 positions in which to place the 2 bars.

$\binom{7}{2}$ ways to do so and $\binom{7}{2}$ ways to split 5$ among 3 people.

Ways to add up $n$ natural numbers to sum to $k$? or

" $k$ from $n$ with replacement where order doesn't matter."

In general, $k$ stars $n-1$ bars.

$$\star\star|\star|\cdots|\star\star.$$

$n+k-1$ positions from which to choose $n-1$ bar positions.

$$\binom{n+k-1}{n-1}$$

## Stars and Bars Poll

**Mark what's correct:**
(A) ways to split 5 dollars among 3: $\binom{7}{2}$
(B) ways to split n dollars among k: $\binom{n+k-1}{k-1}$
(C) ways to split 3 dollars among 5: $\binom{7}{5}$
(D) ways to split 5 dollars among 3: $\binom{7}{5}$

(A),(B),(D) are correct.

## Combinatorial Proofs - 1

A technique to prove identities by counting arguments!

**Theorem:** $\binom{n}{k} = \binom{n}{n-k}$

**Proof:** How many subsets of size $k$? $\binom{n}{k}$

How many subsets of size $k$?
Choose a subset of size $n-k$
and what's left out is a subset of size $k$.
Choosing a subset of size $k$ is same
as choosing $n-k$ elements to not take.
$\implies \binom{n}{n-k}$ subsets of size $k$.

$\square$

## Combinatorial Proofs - 2

**Theorem:** $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.
**Proof:** How many size $k$ subsets of $n+1$? $\binom{n+1}{k}$.

How many size $k$ subsets of $n+1$?
How many contain the first element?
Chose first element, need $k-1$ more from remaining $n$ elements.
$\implies \binom{n}{k-1}$
How many don't contain the first element ?
Need to choose $k$ elements from remaining $n$ elts.
$\implies \binom{n}{k}$

**Sum Rule: size of union of disjoint sets of objects.**
Without and with first element $\rightarrow$ disjoint.

So, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. $\square$

## Binomial Theorem

**Theorem:** $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$

**Proof:** How many subsets of $\{1,\ldots,n\}$?
Construct a subset with sequence of $n$ choices:
element $i$ **is in** or **is not** in the subset: 2 poss.
First rule of counting: $2 \times 2 \cdots \times 2 = 2^n$ subsets.

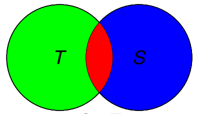
How many subsets of $\{1,\ldots,n\}$?
$\binom{n}{i}$ ways to choose $i$ elts of $\{1,\ldots,n\}$.
Sum over $i$ to get total number of subsets..which is also $2^n$. $\square$

## Simple Inclusion/Exclusion

**Sum Rule: For disjoint sets $S$ and $T$,** $|S \cup T| = |S| + |T|$
Used to reason about all subsets
by adding number of subsets of size 1, 2, 3,...

Also reasoned about subsets that contained
or didn't contain an element. (E.g., first element, first $i$ elements.)

**Inclusion/Exclusion Rule:**
**For any $S$ and $T$,** $|S \cup T| = |S| + |T| - |S \cap T|$.



$S \cap T$

In $T$. $\implies$ $|T|$
In $S$. $\implies$ + $|S|$
Elements in $S \cap T$ are counted twice.
Subtract. $\implies$ $-|S \cap T|$

$|S \cup T| = |S| + |T| - |S \cap T|$

## Simple Inclusion/Exclusion

**Sum Rule: For disjoint sets $S$ and $T$,** $|S \cup T| = |S| + |T|$

**Inclusion/Exclusion Rule: For any $S$ and $T$,**
$|S \cup T| = |S| + |T| - |S \cap T|$.

General version of the above rule (for $n$ sets) in the notes.

**Example:** How many 10-digit numbers (leading 0s OK) have 7 as their first or second digit?

$S$ = numbers with 7 as first digit. $|S| = 10^9$

$T$ = numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T$ = numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

## Summary.

**First rule:** $n_1 \times n_2 \cdots \times n_3$.

$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: $\frac{n!}{(n-k)!}$

**Second rule: when order doesn't matter divide** (when possible)

Sample without replacement and order doesn't matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
"$n$ choose $k$"

**One-to-one rule: equal in number if one-to-one correspondence.**

Sample with replacement and order doesn't matter: $\binom{k+n-1}{n}$.

**Combinatorial Proofs:** Prove identities using counting arguments

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$
**Inclusion/Exclusion Rule: For any $S$ and $T$,**
$|S \cup T| = |S| + |T| - |S \cap T|$.

## Discrete Math for CS... and your future?

Covered many topics: Logic, Proof strategies, Induction, Stable Matching, Graphs, Modular Arithmetic, Polynomials, Countability, Computability, Counting...

Define *precisely*. Understand *properties* of discrete structures. And build from there.

Tools: *formal* reasoning; critical thinking through *proofs*; careful, *rigorous* analysis.

Gives *power to your creativity* and in your pursuits!

....and more to come! Probability Theory!

## Wrapup.

Watch Ed Discussion for Logistics!
Watch Ed Discussion for Advice!

Note your Midterm room assignments!!!

Good Studying and Good Luck!!!