CS70 - Spring 2024

Lecture 15 : March 7

Today: Intro. to Discrete Probability

Q: What is probability?

A: A precise way of talking/reasoning about uncertainty

In Computer Science:

- randomness in data, comms. channels etc.
- probabilistic algorithms

Some Questions we will answer:

1. If we randomly assign 1000 jobs to 1000 processors what's the probable largest load on a processor?

2. In a game of chance at a casino, how likely are we to go bankrupt before we win \$1,000?

3. If a certain medical test comes up negative, what's the chance that the patient has the disease?

4. Can uncertainty sometimes lead to better algorithms?

We always start with a Random Experiment

Example 1 : Toss a fair coin

Possible outcomes : H (Heads)
(Sample space) T (Tails)

Probabilities : H : 1/2
T : 1/2

Sample space : $\Omega = \{H, T\}$

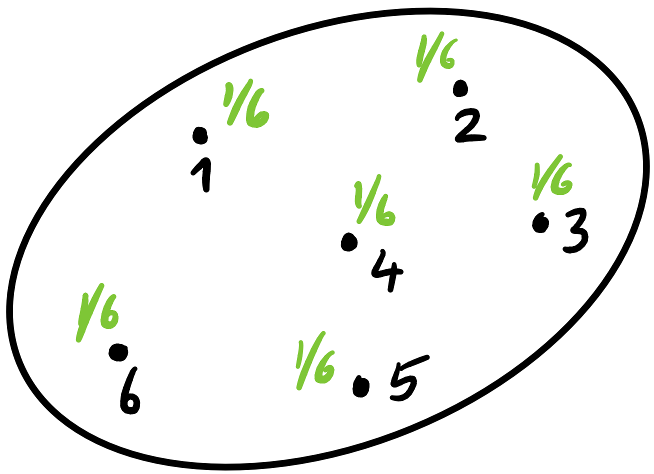
Probabilities : $Pr[H] = Pr[T] = 1/2$

Outcomes/ + Probabilities = $\boxed{\text{Probability Space}}$
Sample Space

Example 2 : Roll a fair (6-sided) die

Sample space : $\Omega = \{1, 2, 3, 4, 5, 6\}$

Probabilities : $\Pr[1] = \Pr[2] = \ldots = \Pr[6] = 1/6$



Example 3 : Toss a biased coin

$\Omega = \{H, T\}$

Probabilities: $\Pr[H] = p \qquad \Pr[T] = 1-p$

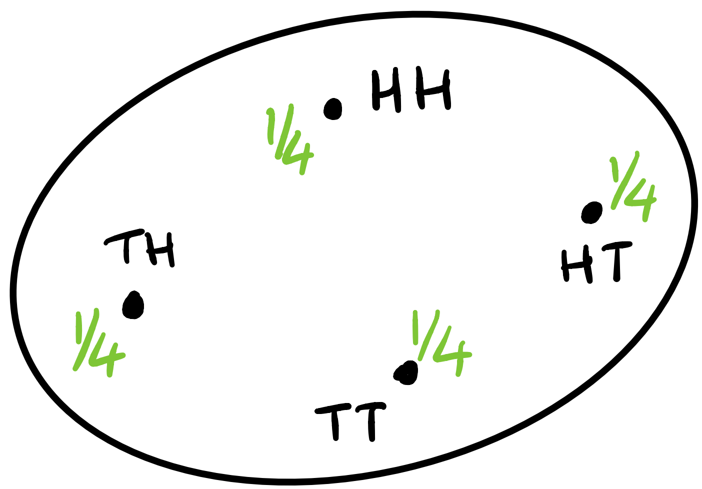
where $0 \le p \le 1$

[$p = 1/2$ is fair coin]

Example 4 : Toss two fair coins

$$\Omega = \{HH, HT, TH, TT\}$$

$$\Pr[HH] = \Pr[HT] = \Pr[TH] = \Pr[TT] = 1/4$$



Example 5: Toss two <u>biased</u> coins, both having Heads probability p

$$\Omega = \{HH, HT, TH, TT\}$$

p^2 $\quad$ $p(1-p)$ $\quad$ $(1-p)p$ $\quad$ $(1-p)^2$

Note: $p^2 + 2p(1-p) + (1-p)^2 = 1 \quad \forall p \in [0,1]$

$$\parallel$$

$$\left(p + (1-p)\right)^2$$

(binomial thm.)

Properties of a Probability Space

Ω : set of outcomes / sample space

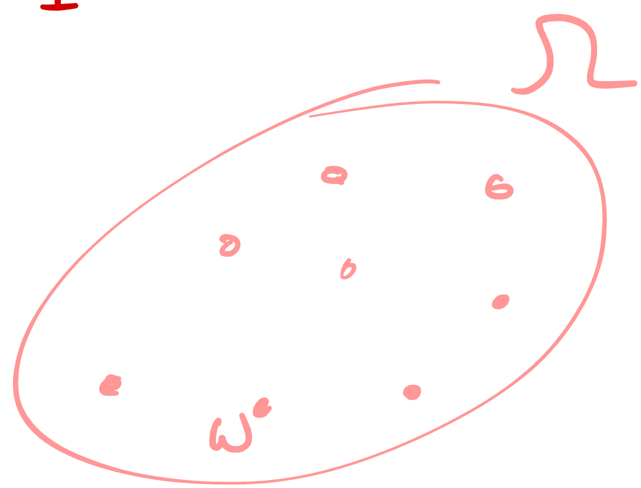
$\omega \in \Omega$: an outcome / sample point

$\Pr[\omega]$: probability of ω $\quad (\forall \omega \in \Omega)$

Probabilities must always satisfy:

(i) $\forall \omega \in \Omega, \quad 0 \leq \Pr[\omega] \leq 1$

(ii) $\displaystyle\sum_{\omega \in \Omega} \Pr[\omega] = 1$



Uniform Probability Space

In a uniform prob. space, all outcomes are equally likely, i.e.,

$$\Pr[\omega] = \frac{1}{|\Omega|} \quad \forall \omega \in \Omega$$

Examples:

- Tossing one (or more) fair coins
- Rolling one (or more) fair dice
- Dealing a poker hand

$\vdots$

Questions about Random Experiments

E.g. Toss two fair coins.
What's the probability exactly one comes up H?

$\Omega = \{HH, HT, TH, TT\}$

Answer: $\Pr[\text{exactly one Head}] = \Pr[HT] + \Pr[TH]$
$= 1/4 + 1/4 = 1/2$

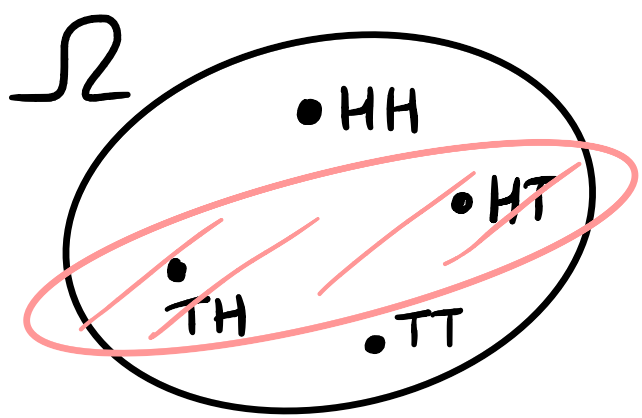
Events

An event, E, is any subset of the sample space, i.e.,

$$E \subseteq \Omega.$$

The probability of E is defined as

$$\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$$



E = exactly one Head

Events in Uniform Prob. Spaces

In a <u>uniform</u> prob. space, $\Pr[\omega] = \frac{1}{|\Omega|} \quad \forall \omega \in \Omega$

and so :

$$\Pr[E] = \sum_{\omega \in E} \Pr[\omega] = \frac{|E|}{|\Omega|}$$

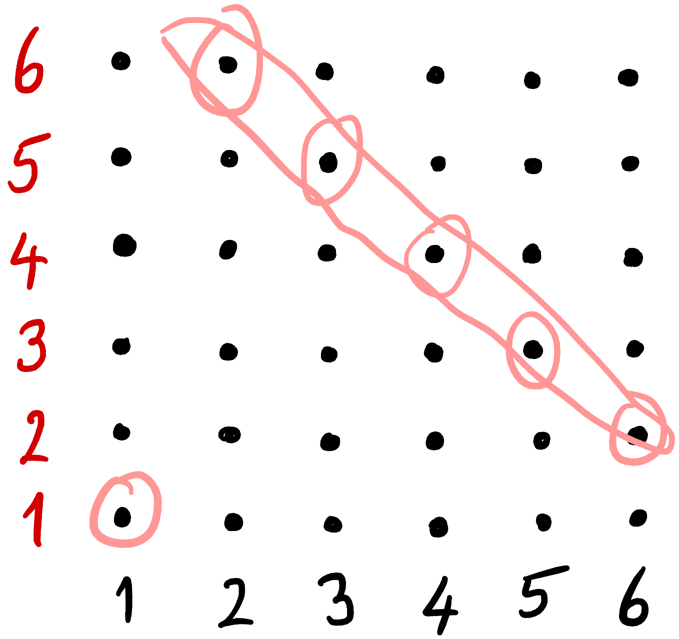
So, in uniform spaces,

Probability = Counting !

<u>Example</u>: Roll two fair dice

What is $\Pr[\text{sum is } 8]$?

$\Omega =$



$|\Omega| = 36$

$\Pr[\omega] = 1/36 \quad \forall \omega \in \Omega$

$\Omega = \left\{ (i,j) : \begin{matrix} 1 \le i \le 6 \\ 1 \le j \le 6 \end{matrix} \right\}$

$\omega = (i,j)$

e.g. $\omega = (3,6)$

Event E_8 = sum is 8 : $\Pr[E_8] = \dfrac{|E_8|}{|\Omega|} = \dfrac{5}{36}$

Event E_2 = sum is 2 : $\Pr[E_2] = \dfrac{|E_2|}{|\Omega|} = \dfrac{1}{36}$

Example: Toss a fair coin 20 times

$\Omega = \{HH\cdots H, HH\cdots HT, \cdots, TT\cdots T\}$ $\quad |\Omega| = 2^{20}$

Q1: Which outcome is more likely?

$\omega_1 = HHHHHHHHHHHHHHHHHHHH$ [20 Heads]

$\omega_2 = THTHHTTHTTTHHTHTHHTH$ [10 Heads]

A1: $\Pr[\omega_1] = 1/2^{20}$

$\Pr[\omega_2] = 1/2^{20}$

Example: Toss a fair coin 20 times

$\Omega = \{HH\cdots H, HH\cdots HT, \cdots, TT\cdots T\}$ $\quad |\Omega| = 2^{20}$

Q1: Which outcome is more likely?

$\omega_1 = HHHHHHHHHHHHHHHHHHHH$ [20 Heads]

$\omega_2 = THTHHTTHTTTHHTHTHHTH$ [10 Heads]

A1: $Pr[\omega_1] =$

$Pr[\omega_2] =$

Q2: Which event is more likely?

$E_{20} =$ 20 Heads

$E_{10} =$ 10 Heads

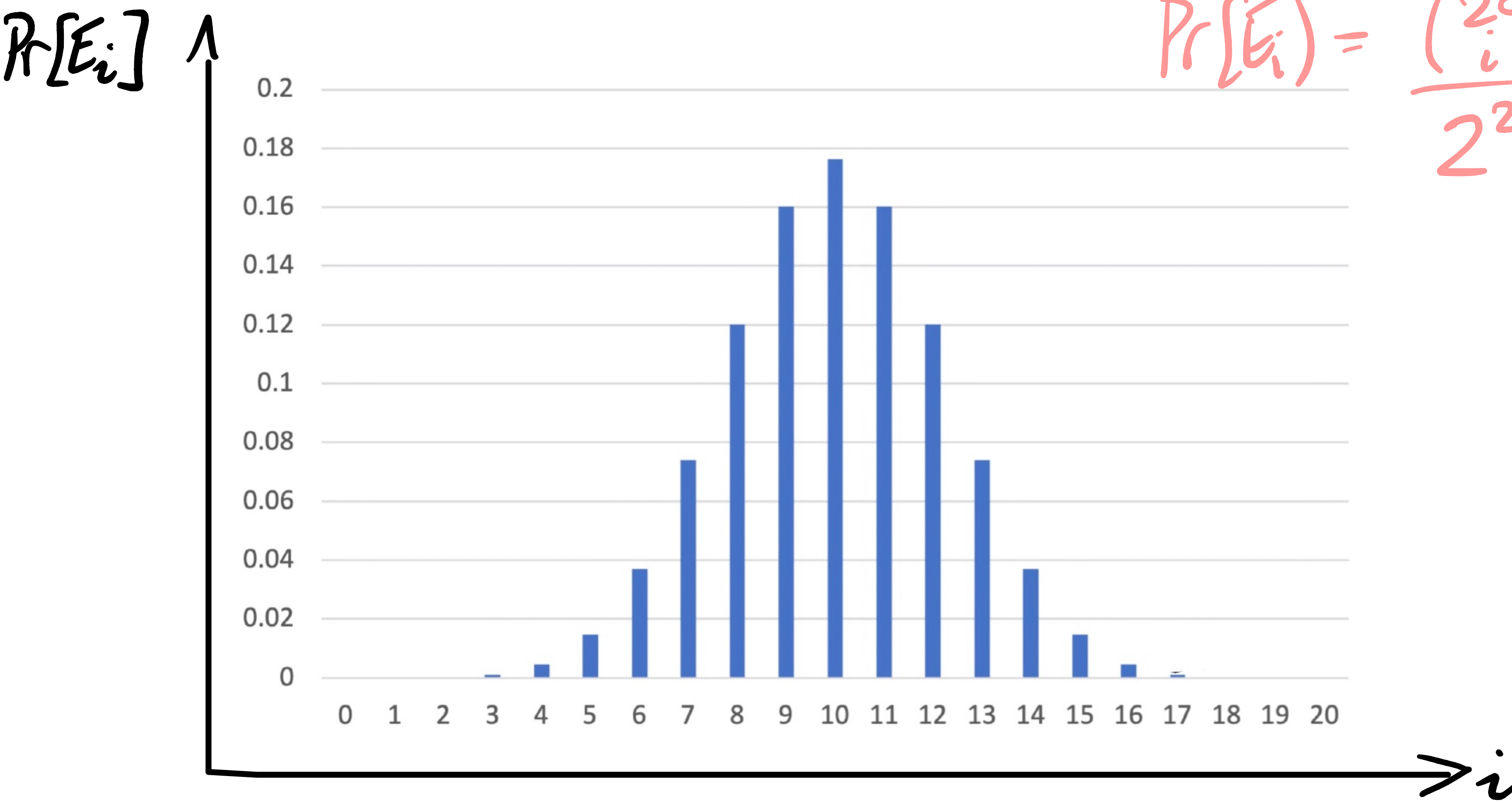
A2: $Pr[E_{20}] = 1/2^{20} \approx 10^{-6}$

$Pr[E_{10}] = \frac{|E_{10}|}{|\Omega|} = \frac{\binom{20}{10}}{2^{20}} = \frac{184{,}756}{2^{20}} \approx 0.176$

Toss a fair coin 20 times

Events E_i = exactly i Heads $\quad (0 \le i \le 20)$

$$\Pr[E_i] = \frac{\binom{20}{i}}{2^{20}}$$



Example: Poker Hands

Ω = set of all possible 5-card poker hands $|\Omega| = \binom{52}{5} \approx 2.6m$

$$\Pr[\omega] = \frac{1}{|\Omega|} \quad \forall \text{ hands } \omega \in \Omega$$

Events:

E_{Ace} = hand contains at least one ace

E_{Flush} = all cards belong to same suit

$E_{StFlush}$ = flush & all cards in sequence

- $\Pr[E_{Ace}] = \dfrac{|E_{Ace}|}{|\Omega|} = 1 - \dfrac{|\overline{E_{Ace}}|}{|\Omega|} = 1 - \dfrac{\binom{48}{5}}{\binom{52}{5}} = 0.34$
- $\Pr[E_{Flush}] = \dfrac{|E_{Flush}|}{|\Omega|} = \dfrac{4 \times \binom{13}{5}}{\binom{52}{5}} = 0.002$
- $\Pr[E_{StFlush}] = \dfrac{|E_{StFlush}|}{|\Omega|} = \dfrac{4 \times 10}{\binom{52}{5}} = \dfrac{40}{\binom{52}{5}} \approx 0.000015$

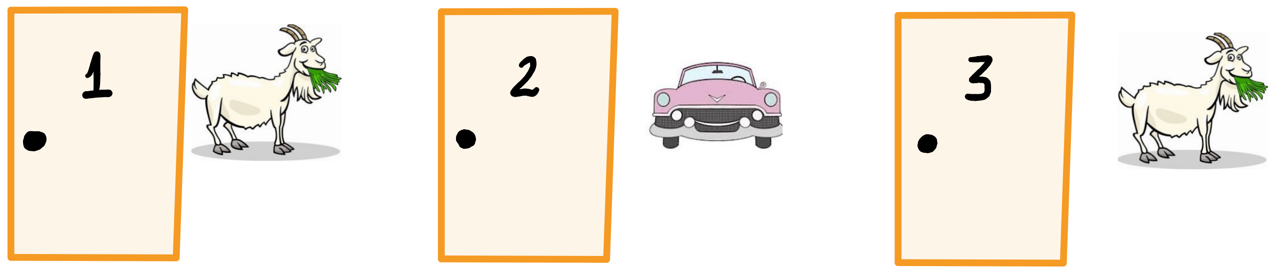
<u>Example</u>: Non-uniform prob. space

Toss two <u>biased</u> coins, Heads prob. p

Event E = exactly one Head

$$\Pr[E] = \underset{p(1-p)}{\Pr[HT]} + \underset{(1-p)p}{\Pr[TH]} = 2p(1-p)$$

Example: Monty Hall Problem



3 doors
1 prize (car)
2 goats

1. Host places prize behind a randomly chosen door
2. You pick some door (say, Door #1)
3. Host opens one of the <u>other</u> doors that has a <u>goat</u>
4. Host offers you the option of sticking or switching doors

Q: What should you do?

"Monty Hall Problem" — inspired by 1970s game show "Let's Make a Deal"

Famously discussed in "Ask Marilyn" column in Parade magazine by Marilyn vos Savant ~1990

Probability space (assuming you initially pick Door #1):

$$\Omega = \{(1,2), (1,3), (2,3), (3,2)\}$$

prize door

door opened by host

$\Pr[(1,2)] = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ (host may open either door)

$\Pr[(1,3)] = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ (——— " ———)

$\Pr[(2,3)] = \frac{1}{3}$ (host <u>must</u> open Door #3)

$\Pr[(3,2)] = \frac{1}{3}$ (——— " ——— #2)

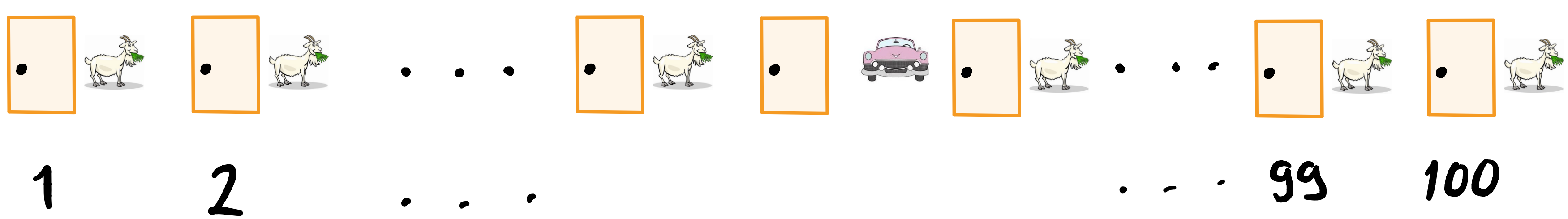
<u>"Sticking" strategy</u>: $\Pr\left[\text{win by sticking}\right] = 1/6 + 1/6 = 1/3$

<u>"Switching" strategy</u>: $\Pr\left[\text{win by switching}\right] = 1/3 + 1/3 = 2/3$

Notes

1. Illustrates importance of understanding / carefully defining the probability space

2. Think about the game with 100 doors:



- You pick (say) Door #1
- Host opens all but one door (leaving just 2 doors)
- Would you switch?

$\Pr[\text{win by switching}] = 99/100$

Summary

- Definition of a probability space:

  Ω = set of outcomes

  $\Pr[\omega]$ = probability for each $\omega \in \Omega$

- Events $E \subseteq \Omega$

$$\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$$

- Uniform probability space:

$$\Pr[\omega] = \frac{1}{|\Omega|} \quad \forall\, \omega \in \Omega$$

$$\Pr[E] = \frac{|E|}{|\Omega|} \quad \forall\, E \subseteq \Omega$$