# CS70 – Spring 2024

## Lecture 16 – March 12

# Last Lecture:

- Definition of a probability space:

  $\Omega$ = set of outcomes

  $\Pr[\omega]$ = probability for each $\omega \in \Omega$

- Events $E \subseteq \Omega$

$$\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$$

- Uniform probability space:

$$\Pr[\omega] = \frac{1}{|\Omega|} \quad \forall \omega \in \Omega$$

$$\Pr[E] = \frac{|E|}{|\Omega|} \quad \forall E \subseteq \Omega$$

Ref: Note 13

Today:

- Conditional probability
- Intersections & unions of events
- Bayes Rule & inference

Ref: Note 14

# Conditional Probability

Recall: 5-card poker hand

$\rightarrow$ uniform prob. space with $|\Omega| = \binom{52}{5}$

Event $E_{Flush}$ = all five cards of same suit

$$\Pr[E_{Flush}] = \frac{|E_{Flush}|}{|\Omega|} = \frac{4 \times \binom{13}{5}}{\binom{52}{5}} \approx 0.002$$

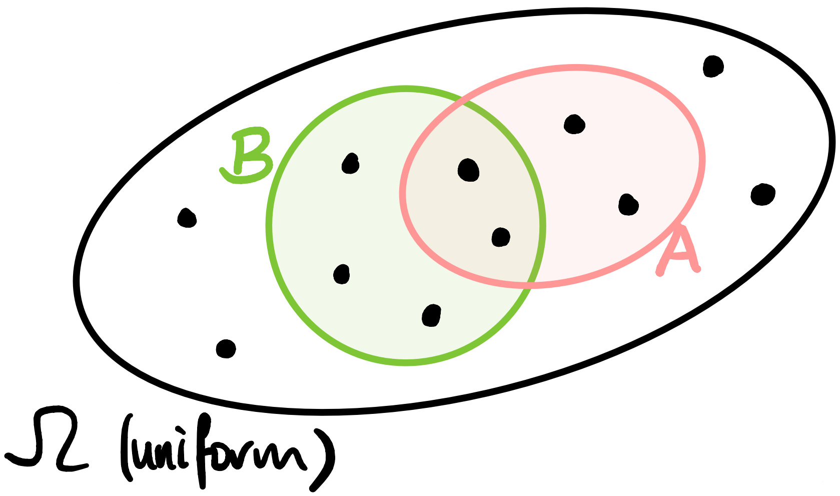
Now suppose your first 4 cards are all $\diamondsuit$

What is now $\Pr[E_{Flush}]$ ?

$$\Pr[E_{Flush} \mid \diamondsuit\diamondsuit\diamondsuit\diamondsuit] = \frac{\#\text{remaining } \diamondsuit}{\#\text{remaining cards}} = \frac{9}{48} \approx 0.19$$

**Defn**: For any events $A, B$ with $Pr[B] > 0$, the <u>conditional probability of $A$ given $B$</u> is

$$Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]}$$




$Pr[A] = \frac{4}{11}$

$Pr[A \mid B] = \boxed{\frac{2}{5}}$

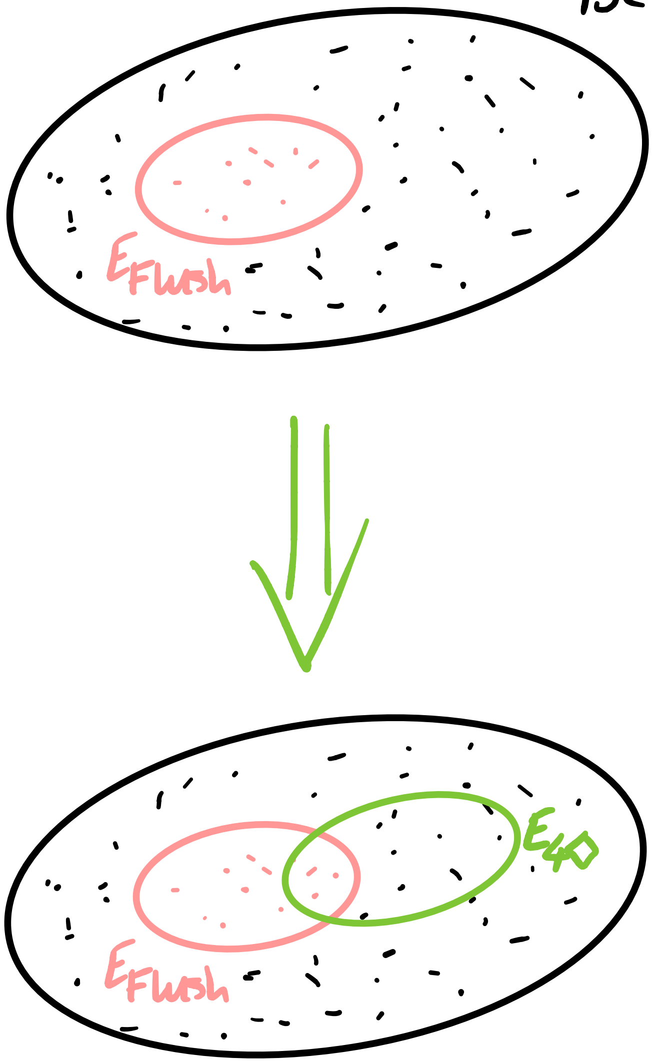
For each sample point $\omega \in B$: $\quad Pr[\omega] \rightarrow \frac{Pr[\omega]}{Pr[B]}$

- - - - - - - - - - $\omega \notin B$: $\quad Pr[\omega] \rightarrow 0$

Then $Pr[A] = \sum_{\omega \in A} Pr[\omega] \longrightarrow \sum_{\omega \in A \cap B} \frac{Pr[\omega]}{Pr[B]} = \frac{Pr[A \cap B]}{Pr[B]}$

# Example: Flush

$|\Omega| = 52!$




$$\Pr[E_{Flush}] = \frac{|E_{Flush}|}{|\Omega|} \simeq 0.002$$

$$\Pr[E_{Flush} \mid E_{4\diamond}] = \frac{|E_{Flush} \cap E_{4\diamond}|}{|E_{4\diamond}|}$$

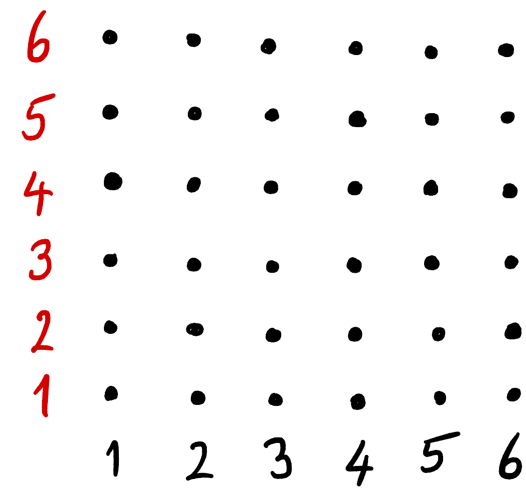
$$= \frac{\binom{13}{5}}{\binom{13}{4} \times 48}$$

$$= \frac{9}{48}$$

# Example: Dice Game

Roll 2 dice – you win if sum is $\geq 9$

$$\Pr[\text{win}] = \frac{|W|}{|\Omega|} = \frac{10}{36} = \frac{5}{18}$$




Define $E_i$ = "red die shows $i$"

$$\Pr[W \mid E_6] = \frac{\Pr[W \cap E_6]}{\Pr[E_6]}$$

$$= \frac{4/36}{1/6} = \boxed{\frac{2}{3}} > \Pr[W]$$

$$\Pr[W \mid E_3] = \frac{\Pr[W \cap E_3]}{\Pr[E_3]}$$

$$= \frac{1/36}{1/6} = \boxed{\frac{1}{6}} < \Pr[W]$$

# Example: Coin Tossing

Toss a fair coin 20 times

$E_i$ = "ith toss comes up Heads"

$\Pr[E_i] = 1/2 \quad \forall i$

Suppose the first 19 tosses all come up Heads

What is now $\Pr[E_{20}]$ ?

$$\Pr[E_{20} \mid E_1 \cap \ldots \cap E_{19}] = \frac{\Pr[E_1 \cap \ldots \cap E_{20}]}{\Pr[E_1 \cap \ldots \cap E_{19}]} = \frac{1/2^{20}}{1/2^{19}}$$

$$= \boxed{\frac{1}{2}} = \Pr[E_{20}]$$

We say that $E_{20}$ is <u>independent</u> of $E_1, \ldots, E_{19}$

# Correlation

We have seen that $\Pr[A|B]$ can be $\left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\}$ $\Pr[A]$

$\Pr[A|B] > \Pr[A] \longrightarrow A, B$ positively correlated

$\Pr[A|B] < \Pr[A] \longrightarrow A, B$ negatively correlated

$\Pr[A|B] = \Pr[A] \longrightarrow A, B$ independent

E.g. Uniform prob. space over US population

$A =$ "gets lung cancer" $\quad B =$ "is a smoker"

$\Pr[A|B] \approx 1.17 \times \Pr[A] \Rightarrow A, B$ positively correlated

Note: This doesn't necessarily imply that smoking causes lung cancer!

# Independence

**Defn:** Events A, B are independent if

$$\Pr[A \mid B] = \Pr[A]$$

or equivalently if

$$\Pr[A \cap B] = \Pr[A] \times \Pr[B]$$

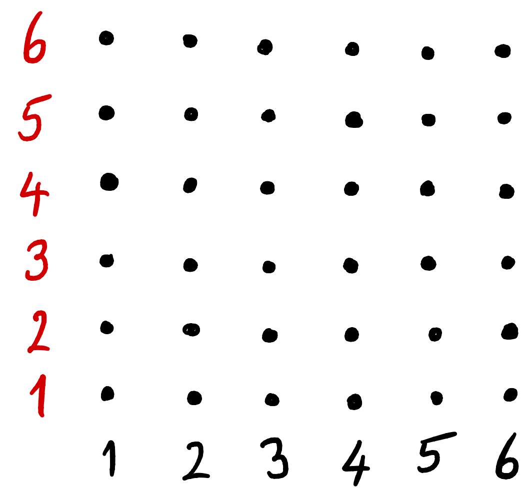
[Equivalent because $\Pr[A \mid B] = \dfrac{\Pr[A \cap B]}{\Pr[B]}$ ]

# Independent or Not ?

1. 20 fair coin tosses

   A = all 20 tosses are H

   B = first 19 tosses are H




2. Roll 2 dice

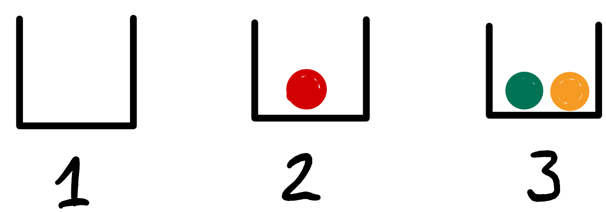
   A = sum is $\geq 10$

   B = first die shows 4

3. Toss 3 balls u.a.r. into 3 bins

   A = bin #1 is empty

   B = bin #2 is empty



# Mutual Independence

**Defn**: Events $A_1, \ldots, A_n$ are <u>mutually independent</u> if for all subsets $I \subseteq \{1, \ldots, n\}$

$$\Pr\left[\bigcap_{i \in I} A_i\right] = \prod_{i \in I} \Pr[A_i]$$

**Example**: 2 fair coin flips

$A_1$: "first flip is H" $\qquad \Pr[A_1] =$

$A_2$: "second flip is H" $\qquad \Pr[A_2] =$

$A_3$: "both flips the same (HH or TT)" $\qquad \Pr[A_3] =$

$A_1, A_2$: independent (obvious)

$A_1, A_3$: $\Pr[A_1 \cap A_3] = \Pr[HH] = 1/4 = \Pr[A_1]\Pr[A_3]$

$A_2, A_3$: same

**BUT**: $\Pr[A_1 \cap A_2 \cap A_3] =$

# Independent Coin Flips

We often use independence to <u>define</u> prob. spaces

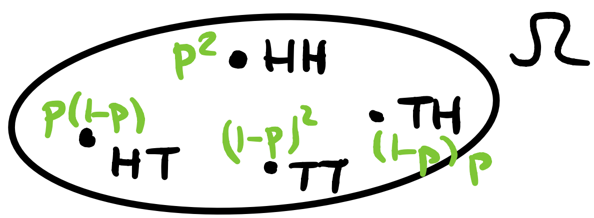
<u>Example</u>: Flipping a biased coin (Heads prob. p) twice

We want the flips to be <u>independent</u>, e.g.,

$$\Pr[HT] = \Pr[\text{1st is } H] \times \underbrace{\Pr[\text{2nd is } H \mid \text{1st is } H]}_{= \Pr[\text{2nd is } H] \text{ (independence)}}$$

$$= p \times (1-p)$$

So we get




More generally, with n flips, for any seq. $\omega$ with $i$ Heads and $n-i$ Tails,

$$\Pr[\omega] = p^i (1-p)^{n-i}$$

# Intersections: Product Rule

Recall: $\Pr[A|B] = \dfrac{\Pr[A \cap B]}{\Pr[B]}$

This implies ...

**Product Rule**: For any events $A, B$

$$\Pr[A \cap B] = \Pr[A|B] \times \Pr[B] = \Pr(B|A] \times \Pr[A]$$

More generally ...

**Product Rule**: For any events $A_1, \ldots, A_n$

$$\Pr[A_1 \cap \ldots \cap A_n] = \Pr[A_1] \times \Pr[A_2 | A_1] \times \ldots \times \Pr[A_n | A_1 \cap \ldots \cap A_{n-1}]$$

**Product Rule**: For any events $A_1, \ldots, A_n$

$$\Pr[A_1 \cap \ldots \cap A_n] = \Pr[A_1] \times \Pr[A_2 | A_1] \times \ldots \times \Pr[A_n | A_1 \cap \ldots \cap A_{n-1}]$$

**Proof**: By induction on $n$.

Base case $n=2$: basic product rule for 2 events

Inductive step $(n \geq 3)$:

$$\Pr[\underbrace{A_1 \cap \ldots \cap A_{n-1}}_{B} \cap A_n] = \Pr[B] \times \Pr[A_n | B]$$

ind. hypothesis

$$= \Pr[A_1] \times \Pr[A_2 | A_1] \times \ldots \times \Pr[A_{n-1} | A_1 \cap \ldots \cap A_{n-2}]$$

$$\times \Pr[A_n | A_1 \cap \ldots \cap A_{n-1}]$$

# Unions of Events

Another dice game:

Roll two fair dice – you win if you roll at least one 6

Pr[roll 6 on one die] = $1/6$

Pr[Win] = Pr[roll 6 on either die] = $1/6 + 1/6 = 1/3$ ?

What if you roll 10 dice?

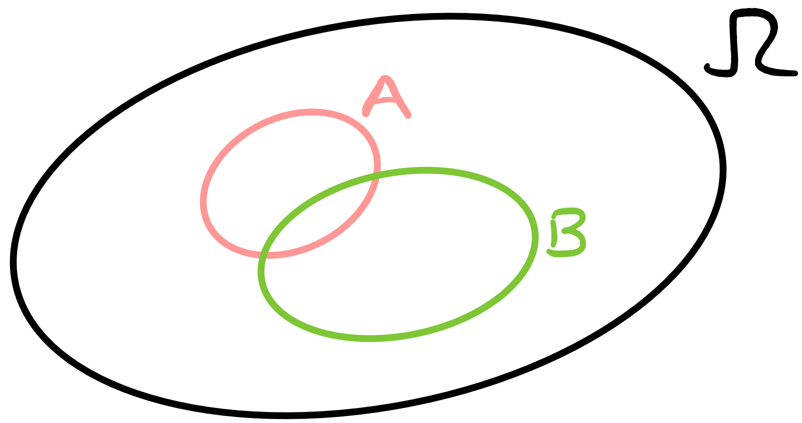
Pr[Win] = $1/6 + \cdots + 1/6 = \frac{10}{6}$ ???

Problem: You may roll more than one 6

Rolling 6's are not disjoint events

**Thm**: For any events A, B

$$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$$




**Proof**: $\Pr[A \cup B] = \sum_{\omega \in A \cup B} \Pr[\omega]$

$$= \sum_{\omega \in A} \Pr[\omega] + \sum_{\omega \in B} \Pr[\omega] - \sum_{\omega \in A \cap B} \Pr[\omega]$$

$$= \Pr[A] + \Pr[B] - \Pr[A \cap B]$$

**Note**: If A, B are <u>disjoint</u> ($A \cap B = \emptyset$) then $\Pr[A \cup B] = \Pr[A] + \Pr[B]$

## Example :

Another dice game :

Roll two fair dice - you win if you roll at least one 6

$\Pr[\text{roll 6 on one die}] = 1/6$

$A$ = roll 6 on first die

$B$ = roll 6 on second die

$$\Pr[\text{Win}] = \Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$$
$$= 1/6 + 1/6 - 1/36$$
$$= 11/36$$

# Inclusion-Exclusion

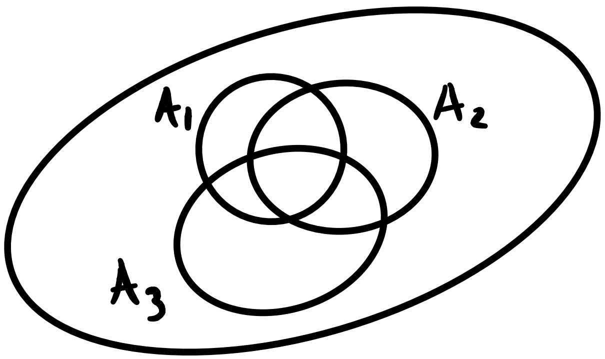
More generally, for any events $A_1, \ldots, A_n$

$$\Pr[A_1 \cup \ldots \cup A_n] = \sum_{i=1}^{n} \Pr[A_i] - \sum_{i<j} \Pr[A_i \cap A_j]$$

$$+ \sum_{i<j<k} \Pr[A_i \cap A_j \cap A_k]$$

$$- \quad \cdots$$

$$\pm \Pr[A_1 \cap \cdots \cap A_n]$$




Proof: See inclusion-exclusion under "Counting"

# Union Bound

**Thm:** For any events $A_1, \ldots, A_n$

$$\Pr[A_1 \cup \ldots \cup A_n] \leq \Pr[A_1] + \ldots + \Pr[A_n]$$

**Proof:** $$\Pr\left[\bigcup_i A_i\right] = \sum_{\omega \in \cup A_i} \Pr[\omega]$$

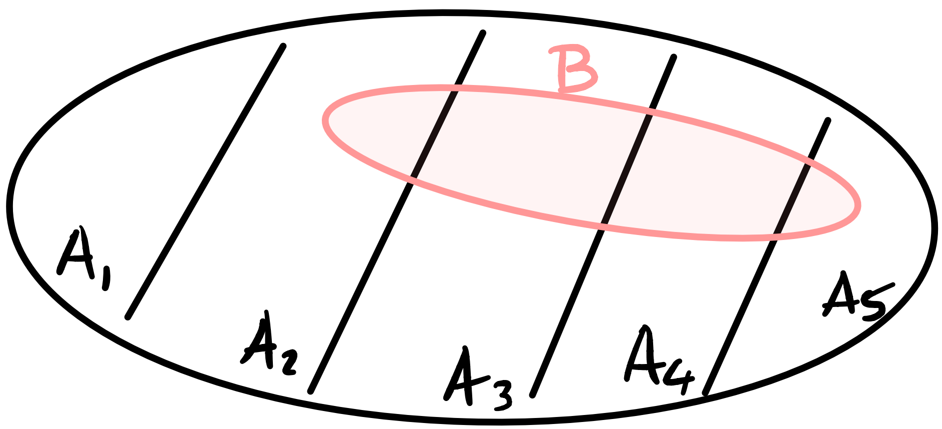
$$\leq \sum_{\omega \in A_1} \Pr[\omega] + \ldots + \sum_{\omega \in A_n} \Pr[\omega]$$

**Later:** We will see how useful this very simple upper bound can be!

# Law of Total Probability

If $A_1, \dots, A_n$ are pairwise disjoint ($A_i \cap A_j = \emptyset \ \forall i \neq j$) and $A_1 \cup \dots \cup A_n = \Omega$, then for any event $B$

$$\Pr[B] = \sum_{i=1}^{n} \Pr[B \cap A_i]$$




Proof: The events $B \cap A_i$ are pairwise disjoint and $B = \bigcup_i (B \cap A_i)$

**Bayes Rule**: For any events $A$, $B$ with $\Pr[A]>0$, $\Pr[B]>0$, we have

$$\Pr[A|B] = \frac{\Pr[B|A]\Pr[A]}{\Pr[B]}$$

**Proof**: Statement is equivalent to

$$\Pr[A|B]\Pr[B] = \Pr[B|A]\Pr[A]$$

This is true because both sides $= \Pr[A \cap B]$

Bayes rule allows us to "flip the conditioning around", from $\Pr[B|A]$ to $\Pr[A|B]$

**Example 1** : Two coins, Heads probs. $p = 1/2$ and $p = 3/5$

- pick a coin u.a.r. ("uniformly at random")
- flip the chosen coin

Suppose the flipped coin comes up Heads
What is the prob. we picked the <u>biased</u> coin ?

$A$ = "picked biased coin"

$B$ = "coin comes up Heads"

We know: $\Pr[A] = 1/2$

$\Pr[B \mid A] = 3/5$ $\quad$ $\Pr[B \mid \bar{A}] = 1/2$

**Goal**: Compute $\Pr[A \mid B]$

$A$ = "picked biased coin"

$B$ = "coin comes up Heads"

We know: $\Pr[A] = 1/2$

$\Pr[B|A] = 3/5 \qquad \Pr[B|\bar{A}] = 1/2$

<u>Goal</u>: Compute $\Pr[A|B]$

<u>Bayes Rule</u>: $\Pr[A|B] = \dfrac{\Pr[B|A]\Pr[A]}{\Pr[B]} = \dfrac{3/5 \times 1/2}{\Pr[B]} = \dfrac{3/10}{\Pr[B]}$

What is $\Pr[B]$ ?

<u>Total Probability</u>: $\Pr[B] = \Pr[B|A]\Pr[A] + \Pr[B|\bar{A}]\Pr[\bar{A}]$

$= \left(\frac{3}{5} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2}\right) = 11/20$

So $\Pr[A|B] = \dfrac{3/10}{11/20} = \boxed{6/11}$

# Updated Bayes Rule

$$\Pr[A|B] = \frac{\Pr[B|A]\Pr[A]}{\underbrace{\Pr[B|A]\Pr[A] + \Pr[B|\bar{A}]\Pr[\bar{A}]}_{=\Pr[B]}}$$

More generally:

$$\Pr[A|B] = \frac{\Pr[B|A]\Pr[A]}{\underbrace{\sum_i \Pr[B|A_i]\Pr[A_i]}_{=\Pr[B]}}$$

where $A_1 \ldots A_n$ partitions $\Omega$

E.g. 3 possible "Go" opponents, one chosen uniformly:

Opp. #1 wins w. prob. 90%
Opp. #2 - - - - - - - 60%
Opp. #3 - - - - - 20%

$$\Pr[\text{you lose}] = \left(\tfrac{1}{3} \times 0.9\right) + \left(\tfrac{1}{3} \times 0.6\right) + \left(\tfrac{1}{3} \times 0.2\right)$$

$$\approx \boxed{0.57}$$

## Example 2 : Medical Testing

Some disease affects 0.1% (=0.001) of population

A test has the following efficacy for a random person:

$$\Pr[\text{test positive} \mid \text{sick}] = 0.99$$

$$\Pr[\text{test positive} \mid \text{not sick}] = 0.01$$

false pos/neg rates are both 0.01

Q: A random person arrives & tests positive. What is the likelihood this person is sick?

$$\Pr[\text{pos.} \mid \text{sick}] = 0.99 \qquad \Pr[\text{sick}] = 0.001$$

$$\Pr[\text{pos.} \mid \text{not sick}] = 0.01$$

Q: A random person arrives & tests positive.
What is the likelihood this person is sick?

$Pr[\text{pos.} \mid \text{sick}] = 0.99$

$Pr[\text{pos.} \mid \text{not sick}] = 0.01$

$Pr[\text{sick}] = 0.001$

Bayes:

$$Pr[\text{sick} \mid \text{pos}] = \frac{Pr[\text{pos} \mid \text{sick}]\, Pr[\text{sick}]}{Pr[\text{pos} \mid \text{sick}]\, Pr[\text{sick}] + Pr[\text{pos} \mid \text{not sick}]\, Pr[\text{not sick}]}$$

$$= \frac{0.99 \times 0.001}{(0.99 \times 0.001) + (0.01 \times 0.999)}$$

$$\approx 0.09$$

Not a great test?
Reason: False pos. rate is large compared to % of sick people

# Simpson's Paradox

On-time arrival performance of two airlines:

| | Airline A | | | Airline B | | |
|---|---|---|---|---|---|---|
| | # flights | # on time | % on time | # flights | # on time | % on time |
| L.A. | 600 | 534 | 89% | 200 | 188 | 94% |
| Chicago | 250 | 176 | 70% | 900 | 685 | 76% |
| Total | 850 | 710 | 84% | 1100 | 873 | 79% |

Which airline would you fly $\begin{cases} \text{into L.A.} & ? \\ \text{into Chicago} & ? \\ \text{overall} & ? \end{cases}$

**Explanation**: Airline A has a much higher percentage of its flights into L.A., which has better performance than Chicago.

**Math**: Pick a random flight . . . .

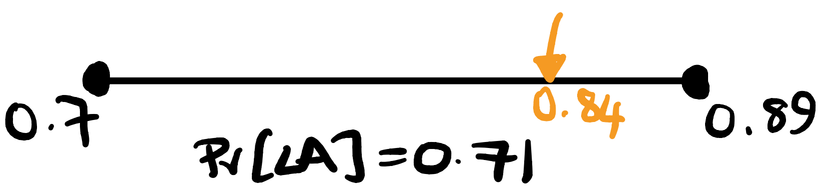
**on Airline A**

$$\Pr[\text{on time} \mid LA] = 0.89$$

$$\Pr[\text{on time} \mid \text{Chicago}] = 0.70$$

$$\Pr[\text{ontime}] = \Pr[\text{ontime} \mid LA]\Pr[LA] + \Pr[\text{ontime} \mid \text{Chic.}]\Pr[\text{Chic.}]$$

$$= \left( (0.89 \times \Pr[LA]) + (0.70 \times \Pr[\text{Chic.}]) \right)$$



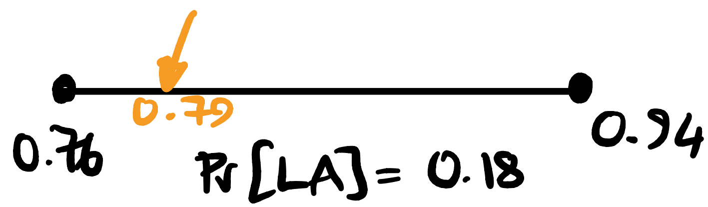

**on Airline B**

$$\Pr[\text{on time} \mid LA] = 0.94$$

$$\Pr[\text{on time} \mid \text{Chic.}] = 0.76$$

$$\Pr[\text{on time}] = \ldots$$

$$= (0.94 \times \Pr[LA]) + (0.70 \times \Pr[\text{Chic}$$



# Summary

- Conditional probability
- Correlation & Independence
- Unions & intersections of events
- Bayes Rule & Total Probability Rule
- Inference ; Simpson's Paradox