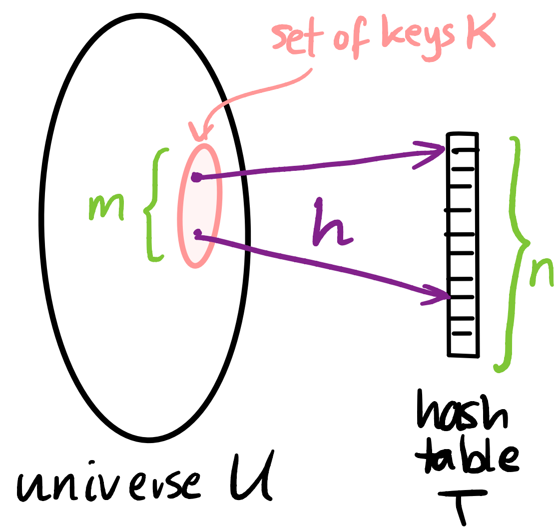
CS70 – Spring 2024

Lecture 18 – March 19

Summary of Last Lecture – all based on $\boxed{\text{balls \& bins}}$

- For a random hash function, to avoid collisions (with good prob.) need size of hash table to be $\approx \boxed{(\text{no. of keys stored})^2}$



- To collect at least one copy of each of n coupons, need to take about $\boxed{n \ln n}$ random samples

- If we randomly distribute n jobs among n processors, the largest load on any processor is likely to be around $\boxed{\dfrac{\ln n}{\ln \ln n}}$

Ideas/techniques are more important than calculations !?

Today

- Random variables (= functions/measurements on probability spaces)
- Distributions
- Expectation
- The Unreasonable Power of Linearity of Expectation

Random Variables

Measurements on probability spaces

$\boxed{\text{Example}}$: Ω = space of 3 fair coin tosses

$= \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Uniform probabilities: $\Pr[\omega] = \frac{1}{8} \quad \forall \omega \in \Omega$

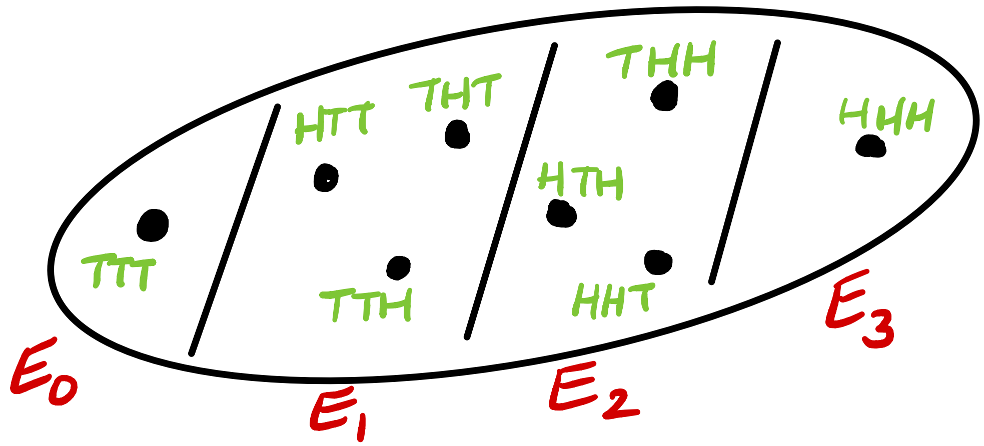
For any $\omega \in \Omega$, let $\boxed{X(\omega) := \text{number of Heads in } \omega}$

$\boxed{\text{Note}}$:

- $X(\omega)$ is a (real) number (actually, a non-neg. integer)
- $X(\omega) \in \{0, 1, 2, 3\}$

For any $\omega \in \Omega$, let $\boxed{X(\omega) := \text{number of Heads in } \omega}$

- For any $i \in \{0,1,2,3\}$, $\boxed{E_i := \{\omega : X(\omega) = i\}}$ is an <u>event</u>
- The events $\{E_i\}$ <u>partition</u> Ω



The collection $\{(i, \Pr[E_i])\}$ is the <u>distribution</u> of X

<u>Here</u>:

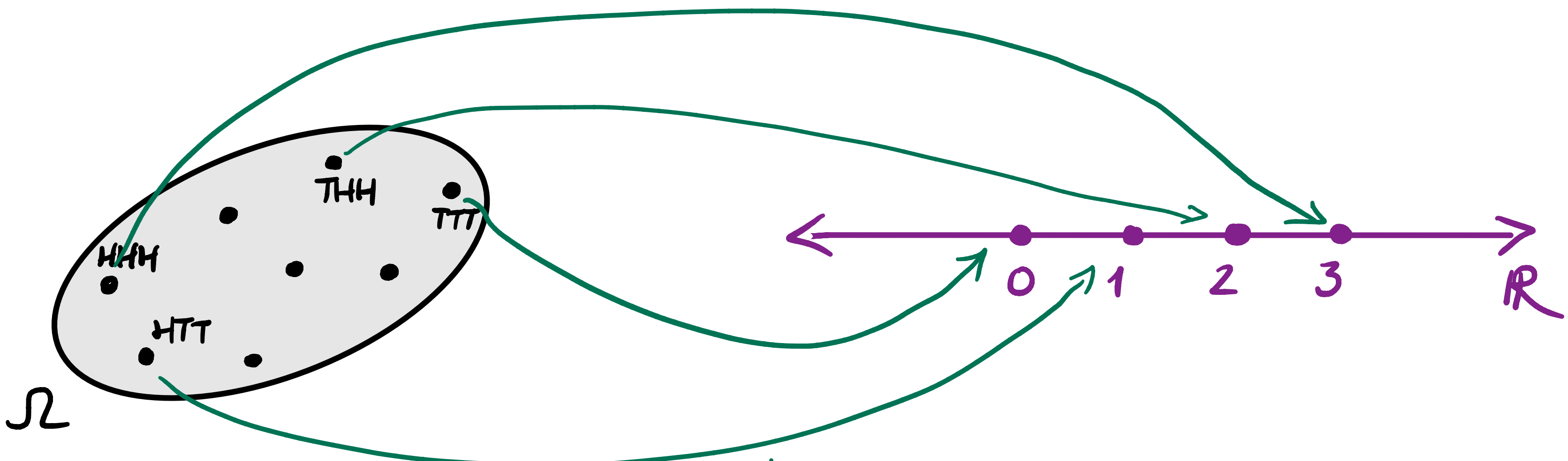
$$\boxed{\begin{aligned} \Pr[X=0] &= 1/8 \\ \Pr[X=1] &= 3/8 \\ \Pr[X=2] &= 3/8 \\ \Pr[X=3] &= 1/8 \end{aligned}}$$

$\leftarrow \Pr[E_0]$
$\leftarrow \Pr[E_1]$
$\leftarrow \Pr[E_2]$
$\leftarrow \Pr[E_3]$

Random Variable: Definition

Defn: A random variable on a prob. space Ω is a function $X : \Omega \to \mathbb{R}$

I.e., X assigns a real value $X(\omega)$ to each $\omega \in \Omega$



$X = \#$ Heads

Defn: The distribution of a (discrete) r.v. X is:

- the set of possible values for X
- for each possible value a the probability $\Pr[X=a]$

Check: For any r.v. X with set of possible values $\mathcal{A}$, we have

$$\sum_{a \in \mathcal{A}} \Pr[X=a] = 1$$

Proof: $\displaystyle \sum_{a \in \mathcal{A}} \Pr[X=a] = \sum_{a \in \mathcal{A}} \left(\sum_{\omega : X(\omega)=a} \Pr[\omega] \right)$

$\displaystyle = \sum_{\omega \in \Omega} \Pr[\omega] = 1$ $\square$

since $\forall \omega \in \Omega$ $\exists$ unique $a \in \mathcal{A}$ s.t. $X(\omega) = a$

Examples

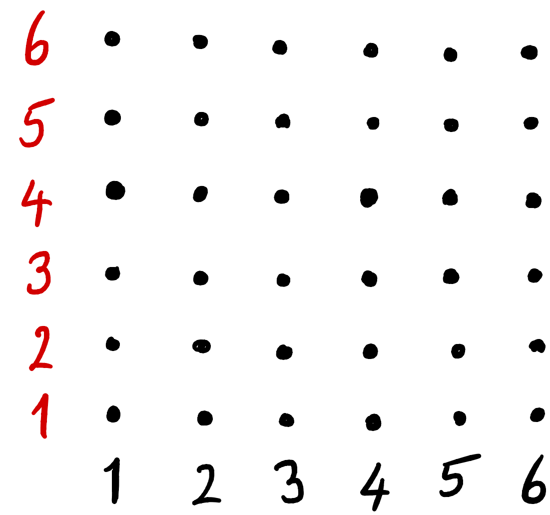
1. Roll 2 fair dice

$X =$ sum of scores on dice

$\Omega = \{(i,j) : 1 \le i,j \le 6\}$ $\quad |\Omega| = 36$

$\boxed{X(i,j) = i+j}$ $\quad X(\omega) \in \{2, 3, \ldots, 11, 12\}$

Distribution of X



$\Pr[X=2] = 1/36$

$\Pr[X=3] = 2/36 = 1/18$

$\Pr[X=4] = 3/36 = 1/12$

$\vdots$

$\Pr[X=7] = 6/36 = \cancel{1/2}\ 1/6$

$\vdots$

$\Pr[X=12] = 1/36$

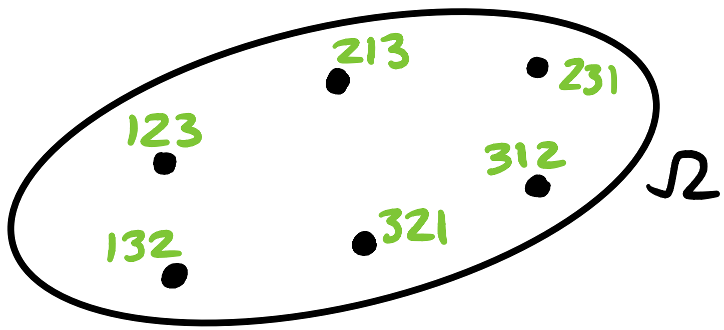
2. Random Permutations

Collect the IDs of n students
Redistribute them randomly (one per student)

Ω = set of permutations of n items

$|\Omega| = n!$

E.g. for $n=3$



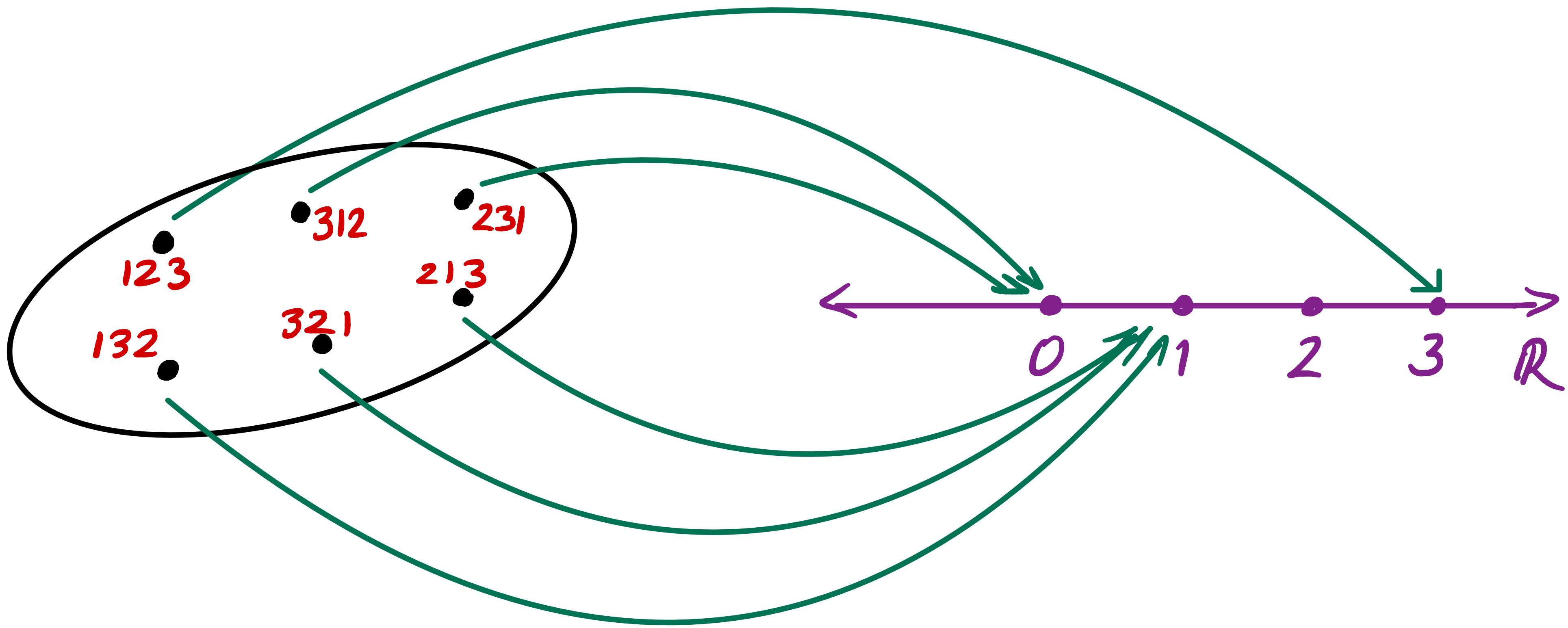
$|\Omega| = 3! = 6$

Uniform probability space: $\Pr[\omega] = \frac{1}{n!} \quad \forall \omega$

Random variable X = no. of students who get their own ID

a.k.a. "fixed points"

X = no. of fixed points in a random permutation



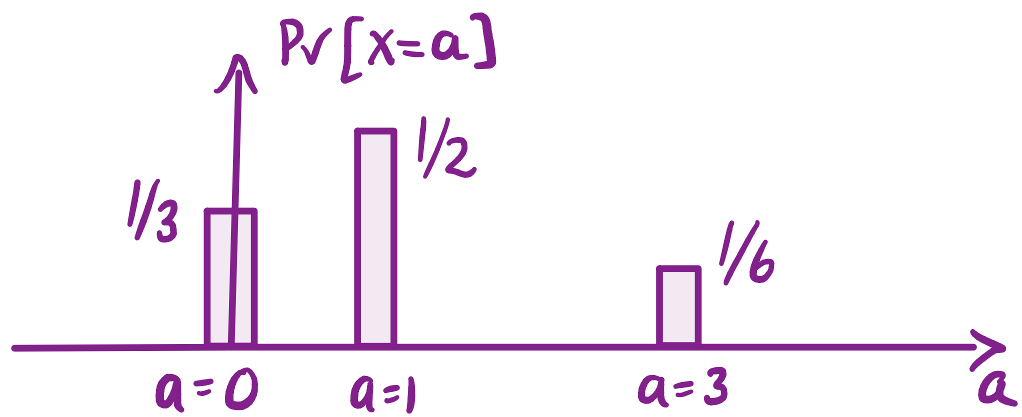
Distribution of X

$\Pr[X=0] = 2/6 = 1/3$

$\Pr[X=1] = 3/6 = 1/2$

$\Pr[X=3] = 1/6 = 1/6$

Histogram



3. Binomial Distribution

Toss n independent biased coins, each having Heads prob. p

$\Omega = \{H,T\}^n$ (= all strings of length n over alphabet $\{H,T\}$)

$$\boxed{\Pr[\omega] = p^i (1-p)^{n-i}}$$ where i = no. of Heads in ω

Random variable X = no. of Heads $X \in \{0,1,\dots,n\}$

What is the distribution of X ?

$$\boxed{\Pr[X=i] = \binom{n}{i} p^i (1-p)^{n-i}}$$

$\dots H \dots HH \dots$

$$\sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = 1$$

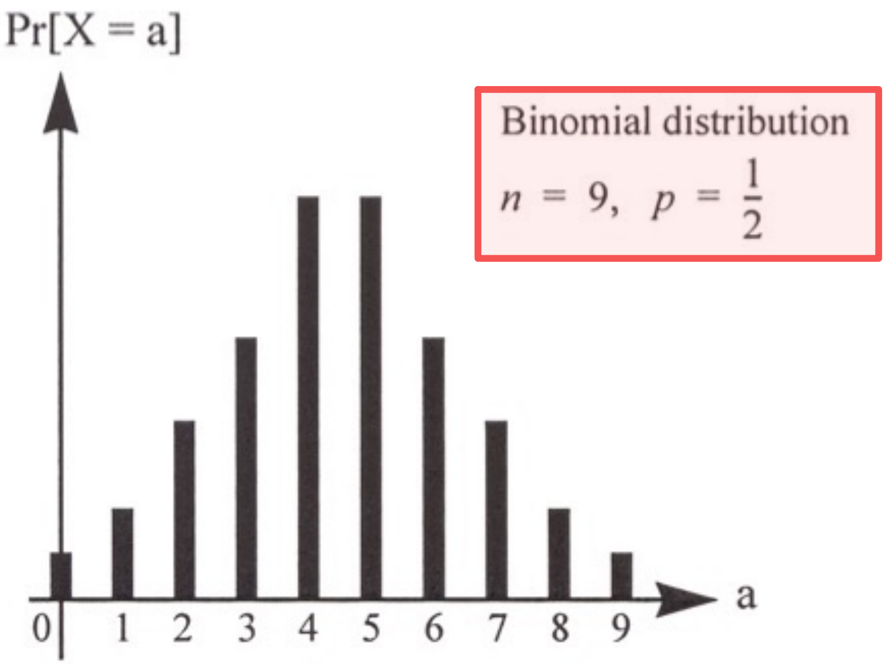
$$\text{"} \quad (p + (1-p))^n$$

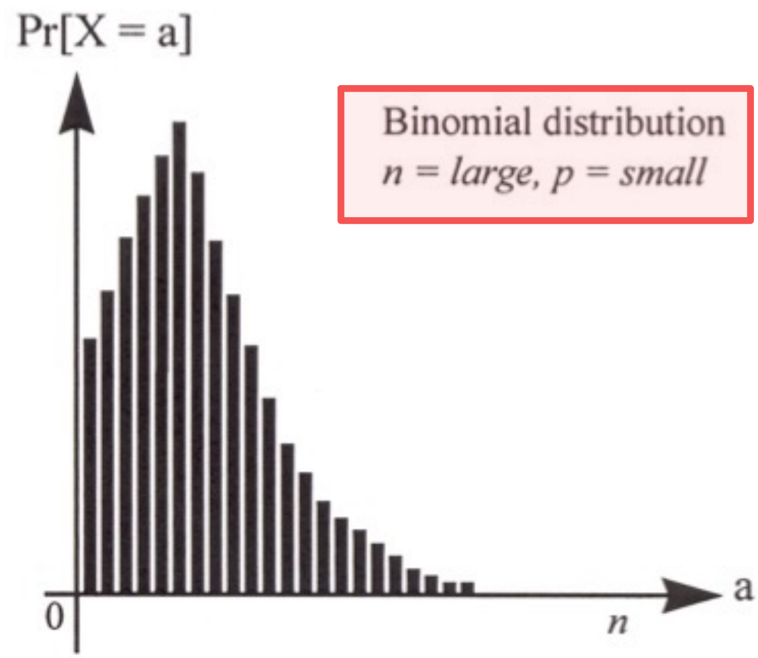
$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

We say X has binomial distribution with parameters n, p

$$\boxed{X \sim \mathrm{Bin}(n,p)}$$

Pictures of Bin(n, p)





4. Hypergeometric Distribution

Deal a 5-card poker hand $\quad |\Omega| = \binom{52}{5}$

R.v. X = no. of hearts in your hand

$X \in \{0, 1, 2, 3, 4, 5\}$

Distribution:

$$\Pr[X=0] = \frac{\binom{39}{5}}{\binom{52}{5}}$$

$$\Pr[X=1] = \frac{\binom{13}{1} \times \binom{39}{4}}{\binom{52}{5}}$$

$\vdots$

$$\Pr[X=k] = \frac{\binom{13}{k}\binom{39}{5-k}}{\binom{52}{5}}$$

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$$\Pr[X=k] = \frac{\binom{13}{k}\binom{39}{5-k}}{\binom{52}{5}}$$

Note: $\sum_{k=0}^{n} \binom{B}{k}\binom{N-B}{n-k} = \binom{N}{n}$!

More generally:

- box of N balls, B black, rest white
- draw n balls w.o. replacement
- X = # of black balls drawn

$$\Pr[X=k] = \frac{\binom{B}{k}\binom{N-B}{n-k}}{\binom{N}{n}}$$

Hypergeometric distribution, parameters (N, B, n)

Joint Distributions

Defn: The joint distribution of two r.v.'s X, Y on the same prob. space is the set

$$\boxed{\{(a, b, \Pr[X=a, Y=b]) : a \in \mathcal{A}, b \in \mathcal{B}\}}$$

where $\mathcal{A}, \mathcal{B}$ are the possible values of X, Y resp.

The marginal distribution of X is given by

$$\Pr[X=a] = \sum_{b \in \mathcal{B}} \Pr[X=a, Y=b]$$

X, Y are independent if

$$\Pr[X=a, Y=b] = \Pr[X=a] \times \Pr[Y=b] \quad \forall a, b$$

Joint Distributions

Example: Throw two fair dice

Random variables:

$$\boxed{\begin{array}{l} X = \text{score on first die} \\ Y = \text{— " — second — " —} \\ Z = \text{sum of scores} \end{array}}$$

$\Pr[X=3, Y=5] = 1/36$

$\Pr[X=3, Z=9] = 1/36$

X, Y independent ?

X, Z independent ?

Expectation (= mean/average)

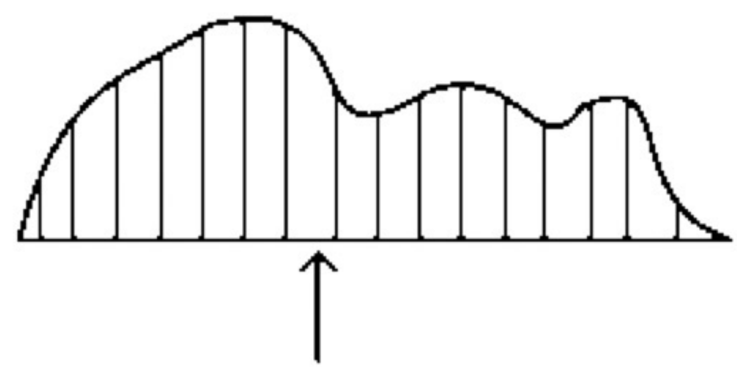
Simplest quantity that summarises the distribution of a r.v.

Defn: The expectation of a (discrete) r.v. X is

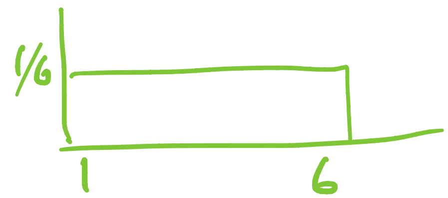
$$\boxed{E[X] := \sum_{a \in \mathcal{A}} a \times \Pr[X=a]}$$

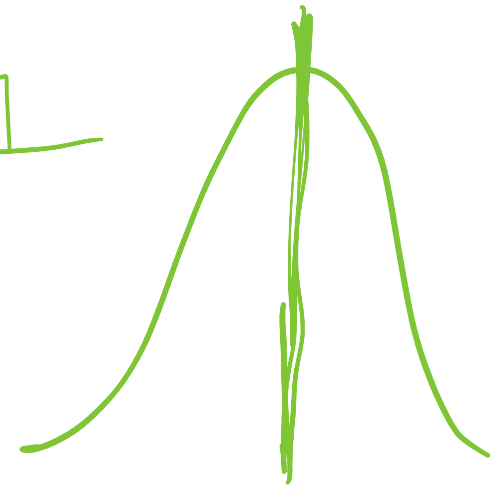
where $\mathcal{A}$ is the set of possible values of X

$E[X]$ measures the "center of mass" of the distribution



Expectation: Examples

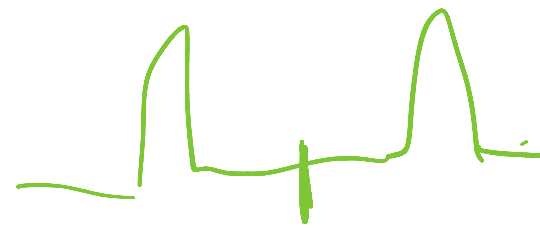




1. X = score on <u>one</u> fair die

$$E[X] = \left(\tfrac{1}{6}\times 1\right) + \left(\tfrac{1}{6}\times 2\right) + \cdots + \left(\tfrac{1}{6}\times 6\right)$$

$$= \tfrac{1}{6}\times\left(1+2+\cdots+6\right) = \boxed{\tfrac{7}{2}}$$

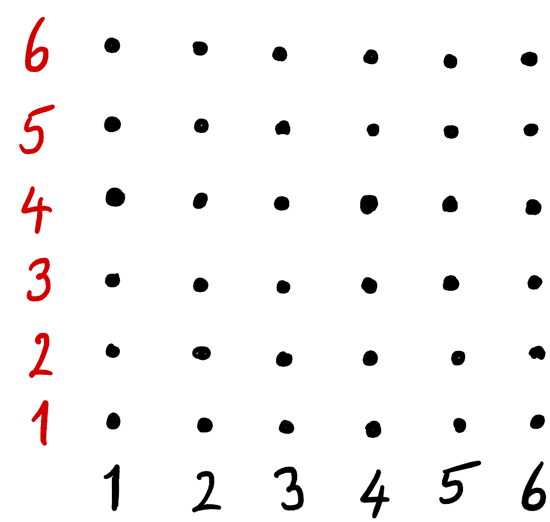


1'. Y = sum of scores on <u>two</u> fair dice

$$E[Y] = \left(\tfrac{1}{36}\times 2\right) + \left(\tfrac{2}{36}\times 3\right) + \left(\tfrac{3}{36}\times 4\right) + \cdots + \left(\tfrac{1}{36}\times 12\right)$$

$$= \ldots$$

$$= \boxed{7}$$



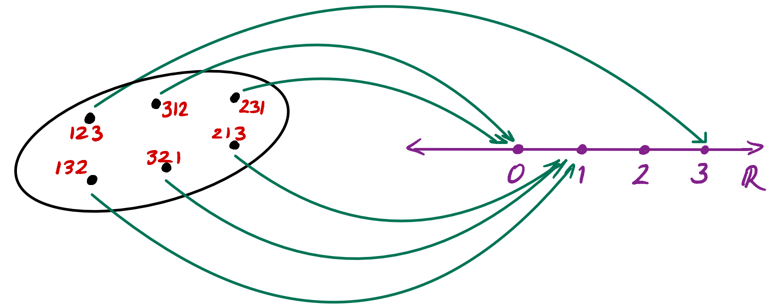
2. X = no. of fixed points in a random permutation $(n=3)$

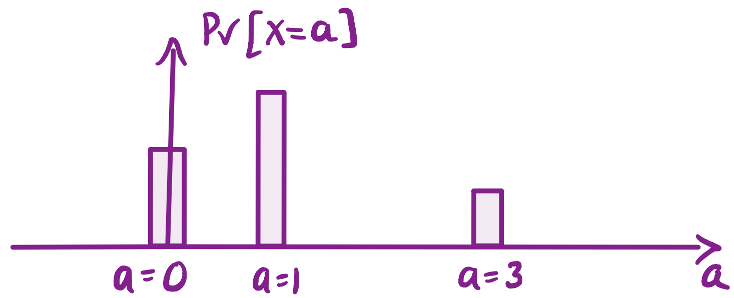
Distribution of X

$Pr[X=0] = 2/6 = 1/3$

$Pr[X=1] = 3/6 = 1/2$

$Pr[X=3] = 1/6 = 1/6$





$$E[X] = \left(\tfrac{1}{3}\times 0\right) + \left(\tfrac{1}{2}\times 1\right) + \left(\tfrac{1}{6}\times 3\right) = 0 + \tfrac{1}{2} + \tfrac{1}{2} = \boxed{1}$$

3. Roulette

Roulette wheel : 36 numbers
(18 black / 18 red) plus 0, 00

Bet \$1 on red : win \$1 if red,
lose \$1 if black or green

X = amount won/lost $\qquad X \in \{-1, +1\}$

$$\left.\begin{array}{l} \Pr[X=+1] = \frac{18}{38} \\ \Pr[X=-1] = \frac{20}{38} \end{array}\right\} E[X] = \left(1 \times \frac{18}{38}\right) + \left(-1 \times \frac{20}{38}\right) = \boxed{-\frac{1}{19}}$$

<u>Note</u> : presence of 0, 00 make this game <u>unfair</u>

Linearity of Expectation

Thm: For any random variables X, Y on prob. space Ω,

(i) $\boxed{E[X+Y] = E[X] + E[Y]}$

(ii) $\boxed{E[aX] = aE[X]}$ for constant a

Proof: Note that $E[X] = \sum_{\omega \in \Omega} X(\omega) \times Pr[\omega]$

(i) $E[X+Y] = \sum_{\omega \in \Omega} (X+Y)(\omega) \times Pr[\omega]$

$(f+g)(x) = f(x) + g(x)$

$= \sum_{\omega \in \Omega} X(\omega) \times Pr[\omega] + \sum_{\omega \in \Omega} Y(\omega) \times Pr[\omega]$

$= E[X] + E[Y]$

(ii) Easy exercise

Crucial: Does not assume X, Y are independent!!!

Does not say that $E[XY] = E[X]E[Y]$ or $E[1/X] = 1/E[X]$

Linearity of Expectation : Examples

1. Two fair dice

$X =$ sum of dice rolls

Then $\underbrace{X = X_1 + X_2}_{\forall \omega \in \Omega:\ X(\omega) = X_1(\omega) + X_2(\omega)}$ where $X_1 =$ score on first die

$X_2 =$ - - - - - second - -

$i+j = X(i,j) = X_1(i,j) + X_2(i,j) = i+j$

"i" "j"

So by linearity

$$E[X] = E[X_1 + X_2] = E[X_1] + E[X_2] = \frac{7}{2} + \frac{7}{2} = \boxed{7}$$

Linearity of Expectation : Examples

2. Multiple roulette games

Play roulette 100 times (\$1 stake each time)

X = amount won/lost

$X = X_1 + X_2 + \cdots + X_{100}$ where X_i = amt. won/lost in ith game

Recall: $E[X_i] = -\frac{1}{19} \quad \forall i$

Linearity: $E[X] = \sum_{i=1}^{100} E[X_i] = 100 \times \left(-\frac{1}{19}\right) \approx \boxed{-5.26}$

3. Multiple coin tosses

indicator r.v.

Toss a biased coin (Heads prob. p) n times

$X = \#$ Heads $\qquad X \sim \text{Binomial}(n, p)$

$X = X_1 + \cdots + X_n \quad$ where $\quad X_i = \begin{cases} 1 & \text{if } i\text{th toss Heads} \\ 0 & \text{if } i\text{th toss Tails} \end{cases}$

Note that $E[X_i] = (1 \times p) + (0 \times (1-p)) = p$

Linearity: $E[X] = \sum_{i=1}^{n} E[X_i] = \boxed{n \times p}$

expectation of a Bin(n,p) distribution

4. Balls & Bins

Recall: toss m balls u.a.r. into n bins

R.v. X = # of empty bins

$$X = \sum_{i=1}^{n} X_i \quad \text{where} \quad X_i = \begin{cases} 1 & \text{if bin } i \text{ empty} \\ 0 & \text{---- not empty} \end{cases}$$

Then $E[X_i] = (1 \times \Pr[\text{bin } i \text{ empty}]) + (0 \times \Pr[\text{bin } i \text{ not empty}])$

$= \Pr[\text{bin } i \text{ empty}]$

$= \left(1 - \frac{1}{n}\right)^m$

all m balls must choose a different bin

$\left(1-\frac{1}{n}\right)^n \sim e^{-1}$

Linearity:

$$E[X] = \sum_{i=1}^{n} E[X_i] = \boxed{n\left(1-\frac{1}{n}\right)^m} \simeq ne^{-m/n}$$

E.g. if $m=n$, $E[X] \simeq ne^{-1} \simeq \boxed{0.37n}$

5. Fixed points in a random permutation

General case: n items

$X = \#$ of fixed points [Recall: $E[X] = 1$ for $n = 3$]

$X = \sum_{i=1} X_i$ where $X_i = \begin{cases} 1 & \text{if } i \text{ is a fixed point} \\ 0 & \text{otherwise} \end{cases}$

$E[X_i] = \Pr[i \text{ is a fixed point}] = 1/n$

Linearity: $E[X] = \sum_{i=1}^{n} E[X_i] = n \times \frac{1}{n} = \boxed{1}$

Bottom line: If we collect & redistribute IDs of n people, the expected # who get their own ID is always 1 (indep. of n)