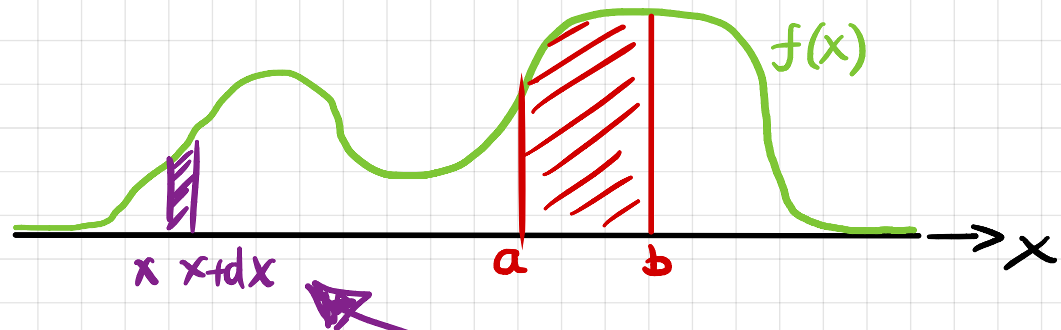
CS70 – Spring 2024

Lecture 23 – April 11

Recap of Previous Lecture

- <u>Continuous</u> r.v. X is described by a prob. <u>density function</u> $f : \mathbb{R} \to \mathbb{R}$ s.t.
  - $f(x) \geq 0 \ \forall x$
  - $\int_{-\infty}^{\infty} f(x)\,dx = 1$
  - $\Pr[a \leq X \leq b] = \int_a^b f(x)\,dx$



- <u>Note</u>: $f(x)$ is probability <u>density</u> per unit length

  $\Pr[x \leq X \leq x+dx] \approx f(x)\,dx$ for infinitesimal dx

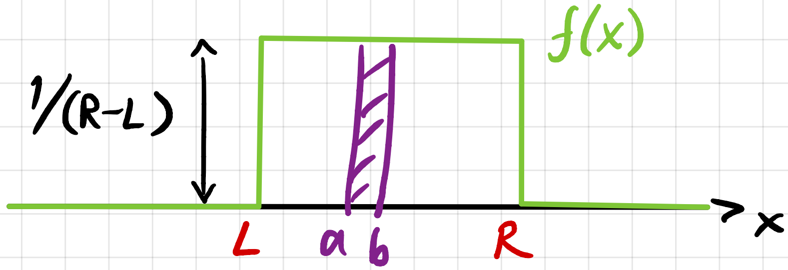
- <u>Cumulative dist. fun.</u> : $F(x) = \Pr[X \leq x] = \int_{-\infty}^{x} f(z)\,dz$

  $f(x) = \frac{d}{dx} F(x)$

Recap (cont.)

- Continuous uniform distribution on $[L, R]$:

$$f(x) = \begin{cases} 0 & x < L \\ 1/(R-L) & L \le x \le R \\ 0 & x > R \end{cases}$$



For $L \le a \le b \le R$: $\quad \Pr[a \le X \le b] = \dfrac{b-a}{R-L}$

- Expectation: $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$

- Variance: $Var(X) = \underbrace{\int_{-\infty}^{\infty} x^2 f(x)\,dx}_{E[X^2]} - E[X]^2$

Recap (cont.)

- **Joint distribution** of r.v.'s X, Y: pdf $f: \mathbb{R}^2 \to \mathbb{R}$

  $f(x,y) \geq 0$

  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

  $\Pr[a \leq X \leq b,\ c \leq Y \leq d] = \int_a^b \int_c^d f(x,y)\,dx\,dy$

- X, Y are **independent** if $\forall a,b,c,d$:

  $\Pr[a \leq X \leq b,\ c \leq Y \leq d] = \Pr[a \leq X \leq b] \cdot \Pr[c \leq Y \leq d]$

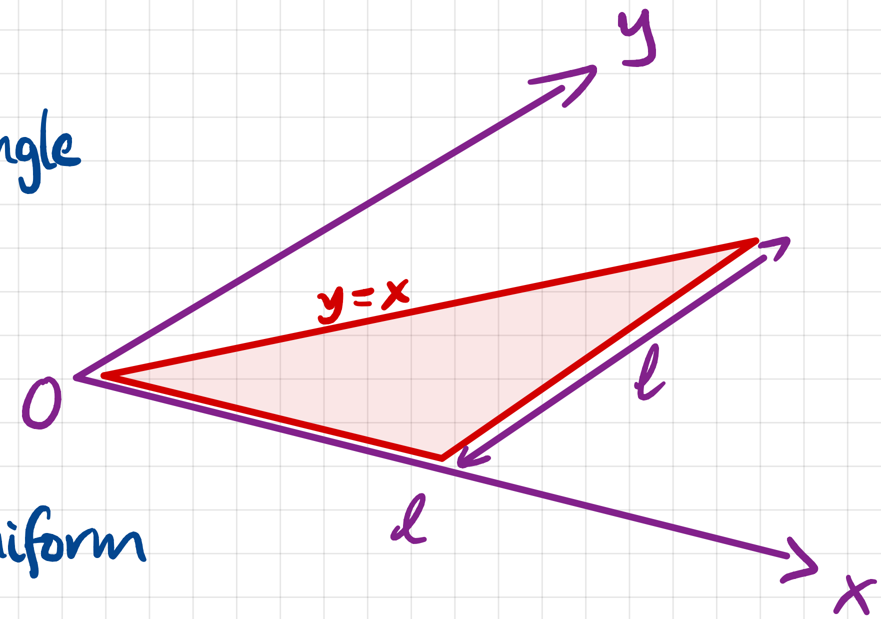
X, Y independent $\Rightarrow$ $f(x,y) = f_X(x)\, f_Y(y)$

From Last Lecture!

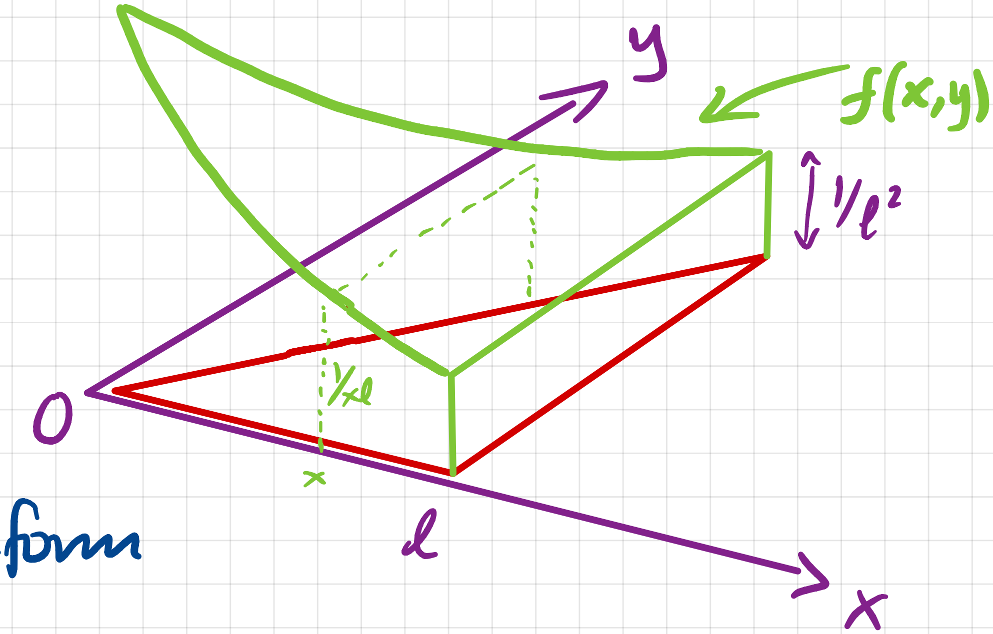
Example: Two-round game

- **Round 1**: You stake \$$\ell$ and win amount X uniform in $[0, \ell]$
- **Round 2**: You stake X and win amount Y uniform in $[0, X]$

winnings from Round 1



- $f(x,y) = 0$ outside red triangle
- Density of X is uniform on $[0, \ell]$
- Given $X = x$, density of Y is uniform on $[0, x]$
- $$f(x,y) = \begin{cases} 1/\ell x & \text{for } (x,y) \in \triangle \\ 0 & \text{otherwise} \end{cases}$$



- $f(x,y) = 0$ outside red $\triangle$
- Density of x is uniform on $[0, \ell]$
- Given x, density of y is uniform on $[0, x]$
- $$f(x,y) = \begin{cases} 1/\ell x & \text{for } (x,y) \in \triangle \\ 0 & \text{otherwise} \end{cases}$$

$$f(x,y) = f_X(x)\, f_{Y|X}(y) = \frac{1}{\ell} \times \frac{1}{x} = \frac{1}{\ell x}$$

Cf. discrete: $\Pr[X=a, Y=b] = \Pr[X=a] \Pr[Y=b \mid X=a]$

<u>Check</u>: $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^{\ell}\int_0^{x} \frac{1}{\ell x}\,dy\,dx = 1$$

<u>Also</u>: $$E[Y] = \int_{-\infty}^{\infty} y\, f(x,y)\,dx\,dy = \frac{\ell}{4} \qquad E[X] = \int_{-\infty}^{\infty} x f(x,y)\,dx\,dy = \frac{\ell}{2}$$

Today

- Exponential distribution
- Normal (Gaussian) distribution
- Central Limit Theorem
  ("averages always look like Gaussians"!)

Exponential Distribution

Continuous - time analog of Geometric distribution

Recall : $X \sim \text{Geom}(p)$

$$\Pr[X=k] = (1-p)^{k-1} p$$

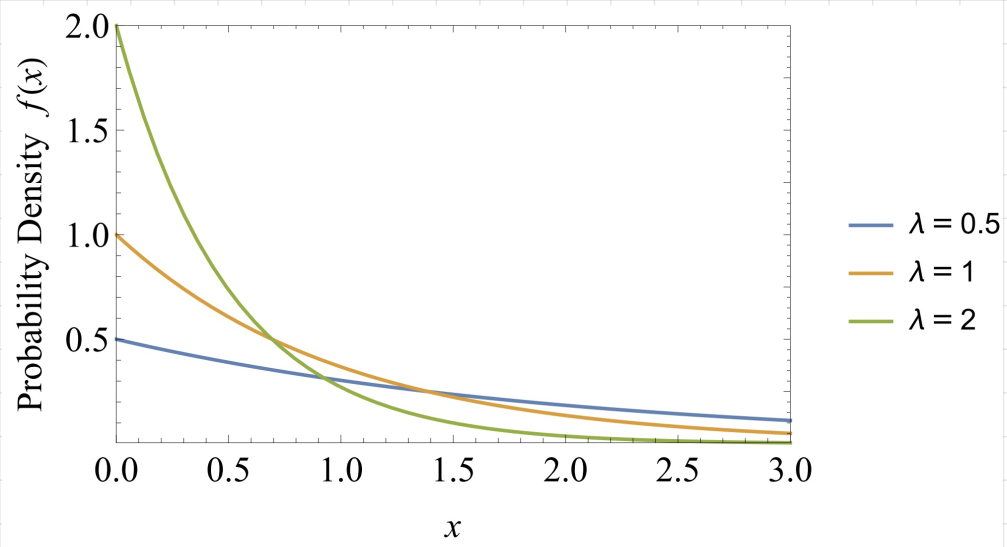
Interpretation : X = no. of trials until the first success where p = success prob.

Exponential distribution measures the time we have to wait until some event happens, given that events happen at fixed rate λ (in continuous time)

Definition: An exponential r.v. X with parameter λ is a continuous r.v. with p.d.f.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write $X \sim Exp(\lambda)$



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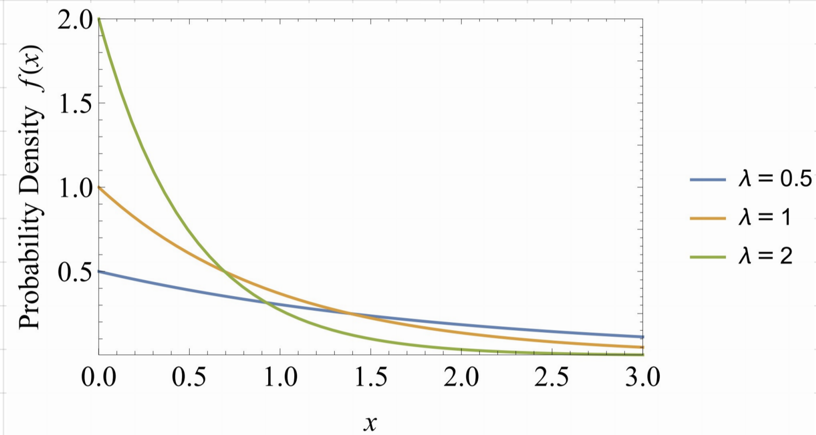


Check p.d.f. conditions:

- $f(x) \geq 0$ ✓
- $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx$

$$= -e^{-\lambda x}\Big|_0^{\infty}$$

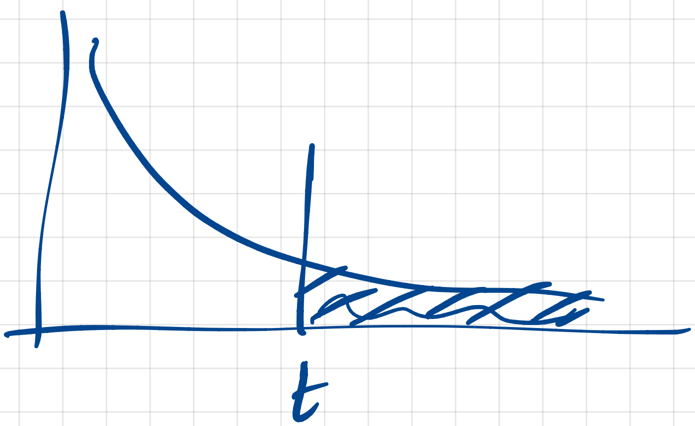
$$= 1 \checkmark$$

Connection with Geometric distribution

$X \sim Exp(\lambda)$

exp. decay, rate λ

Then $Pr[X > t] = \int_t^{\infty} \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_t^{\infty} = \boxed{e^{-\lambda t}}$



Connection with Geometric distribution

$X \sim Exp(\lambda)$

Then $\Pr[X > t] = \int_t^\infty \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_t^\infty = \boxed{e^{-\lambda t}}$

exp. decay, rate λ

Discrete time setting: Sp. we perform one trial every δ seconds, with fixed success prob. λ per unit time

Then # trials until first success is $Y \sim Geom(p)$ where $\boxed{p = \lambda\delta}$. And time until first success is $T = \delta Y$ (seconds)

Connection with Geometric distribution

$X \sim Exp(\lambda)$

exp. decay, rate λ

Then $\Pr[X > t] = \int_t^\infty \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_t^\infty = \boxed{e^{-\lambda t}}$

<u>Discrete time setting</u>: Sp. we perform one trial every δ seconds, with <u>fixed</u> success prob. λ <u>per unit time</u>

Then # trials until first success is $Y \sim Geom(p)$ where $\boxed{p = \lambda\delta}$. And <u>time</u> until first success is $T = \delta Y$ (seconds)

Now let $\delta \to 0$ (so also $p \to 0$)

Then $\Pr[T > t] = \Pr[Y > \frac{t}{\delta}] = (1-p)^{t/\delta} = (1-\lambda\delta)^{t/\delta}$

$\to \boxed{e^{-\lambda t}}$ as $\delta \to 0$ (t, λ fixed)

$(1 - \frac{1}{x})^x \to e^{-x} \quad x \to \infty$

Expectation & Variance

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} \underbrace{x}_{u} \underbrace{\lambda e^{-\lambda x}\,dx}_{dv}$$

$$\left\{\begin{array}{l} u = x \\ du = dx \\ \hline dv = \lambda e^{-\lambda x} dx \\ v = -e^{-\lambda x} \end{array}\right.$$

$$= -xe^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx$$

$$= 0 - \frac{1}{\lambda} e^{-\lambda x}\Big|_0^{\infty} = \boxed{\frac{1}{\lambda}}$$

Recall: "Integration by Parts":

$$\int u\,dv = uv - \int v\,du$$

Expectation & Variance

$$E[X] = \frac{1}{\lambda}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\infty} \underbrace{x^2}_{u} \underbrace{\lambda e^{-\lambda x} dx}_{dv}$$

$$\begin{cases} u = x^2 \\ du = 2x\,dx \\ dv = \lambda e^{-\lambda x} dx \\ v = -e^{-\lambda x} \end{cases}$$

$$= -x^2 e^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} 2x e^{-\lambda x} dx$$

$$= 0 + \frac{2}{\lambda} E[X] = \frac{2}{\lambda^2}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

Recall: "Integration by Parts":

$$\int u\,dv = uv - \int v\,du$$

Normal (a.k.a. Gaussian) Distribution

For any $\mu \in \mathbb{R}$ and $\sigma > 0$, a continuous r.v. X with p.d.f.

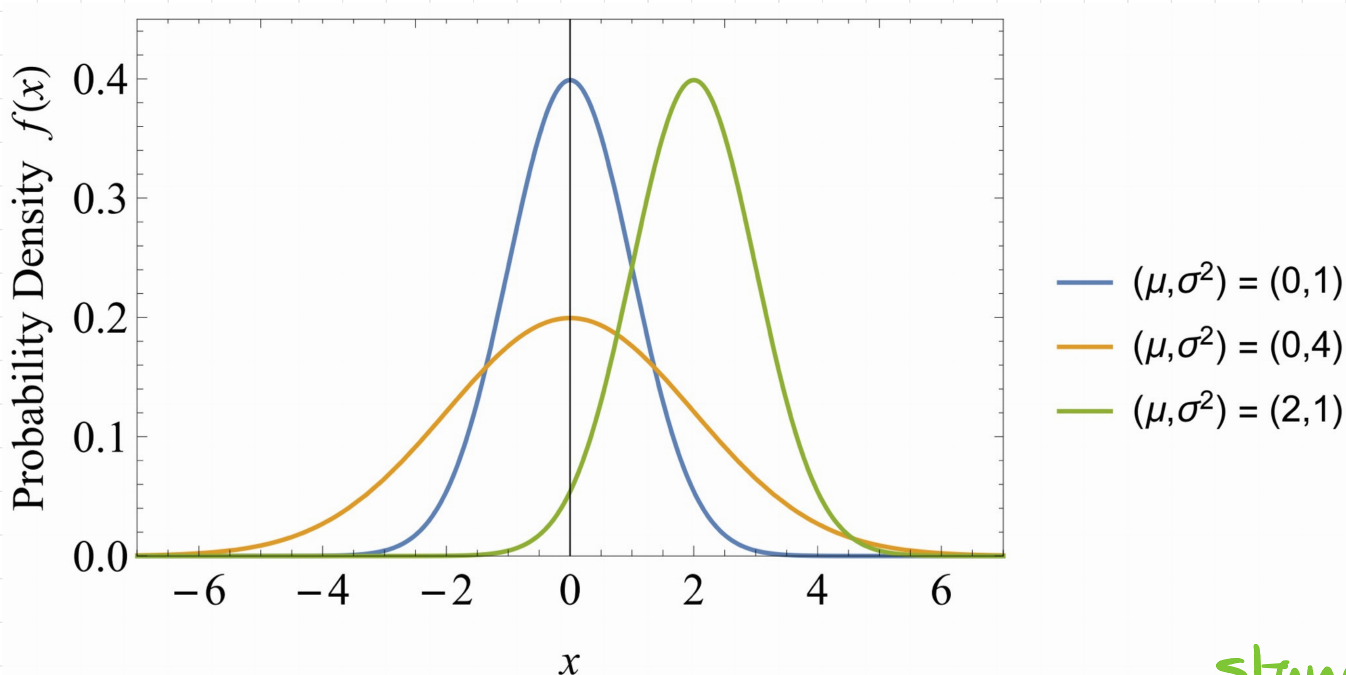
$$\boxed{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}}$$

is a <u>normal r.v.</u> with parameters (μ, σ^2).

We write $\boxed{X \sim N(\mu, \sigma^2)}$

$\mu = 0,\ \sigma = 1 \longrightarrow$ <u>standard</u> normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2}$$



standard integral:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

Check:

- $f(x) \geq 0$ ✓
- $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\, dx = 1$ ✓

Fact: All normal distributions are the same up to shifts and scaling

If $X \sim N(\mu, \sigma^2)$ then $Y = \dfrac{X-\mu}{\sigma} \sim N(0,1)$

Proof: $\Pr[a \leq Y \leq b] = \Pr[\sigma a + \mu \leq X \leq \sigma b + \mu]$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\sigma a+\mu}^{\sigma b+\mu} e^{-(x-\mu)^2/2\sigma^2} \, dx$$

Change of variable: $y = \dfrac{x-\mu}{\sigma}$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{-y^2/2} \cdot \sigma \, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{-y^2/2} \, dy \quad \checkmark$$

Fact: All normal distributions are the same up to shifts and scaling

If $X \sim N(\mu, \sigma^2)$ then $Y = \dfrac{X-\mu}{\sigma} \sim N(0,1)$

Books/internet usually just give c.d.f. of <u>standard</u> normal, i.e., you can look up $\Pr[Y \le c]$ for $Y \sim N(0,1)$

But then if $X \sim N(\mu, \sigma^2)$ you can get

$$\boxed{\Pr[X \le c] = \Pr\left[Y \le \frac{c-\mu}{\sigma}\right]}$$

from table

Expectation & Variance

Suppose $X \sim N(0,1)$ is standard normal

Then p.d.f. is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} x e^{-x^2/2}\,dx + \int_{0}^{\infty} x e^{-x^2/2}\,dx \right] = \boxed{0}$$

$e^{-x^2/2}$ symmetric about 0

Expectation & Variance

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$e^{-x^2/2}$ symmetric about 0

$u = x \qquad dv = x e^{-x^2/2}\,dx$

$du = dx \qquad v = -e^{-x^2/2}$

$$E[X^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[-x e^{-x^2/2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 1 \implies \boxed{Var(X) = 1}$$

Expectation & Variance (cont.)

For $X \sim N(0,1)$: $\quad E[X] = 0 \quad Var(X) = 1$

For $X \sim N(\mu, \sigma^2)$, then $Y = \dfrac{X-\mu}{\sigma} \sim N(0,1)$ so:

$$E[Y] = E\left[\frac{X-\mu}{\sigma}\right] = 0$$

$$\Rightarrow \boxed{E[X] = \mu}$$

$$Var[Y] = Var\left(\frac{X-\mu}{\sigma}\right) = 1$$

$$\Rightarrow \boxed{Var(X) = \sigma^2}$$

This explains notation μ, σ^2

Sum of Independent Normals

Fact: If $X \sim N(0,1)$ and $Y \sim N(0,1)$ are independent, and a, b are constants, then

$$aX + bY \sim N(0, a^2 + b^2)$$

Note: Expectation & variance are obvious from:

$$E[aX+bY] = aE[X] + bE[Y] = 0$$

$$\mathrm{Var}(aX+bY) = \mathrm{Var}(aX) + \mathrm{Var}(bY) \qquad \text{[independent !]}$$

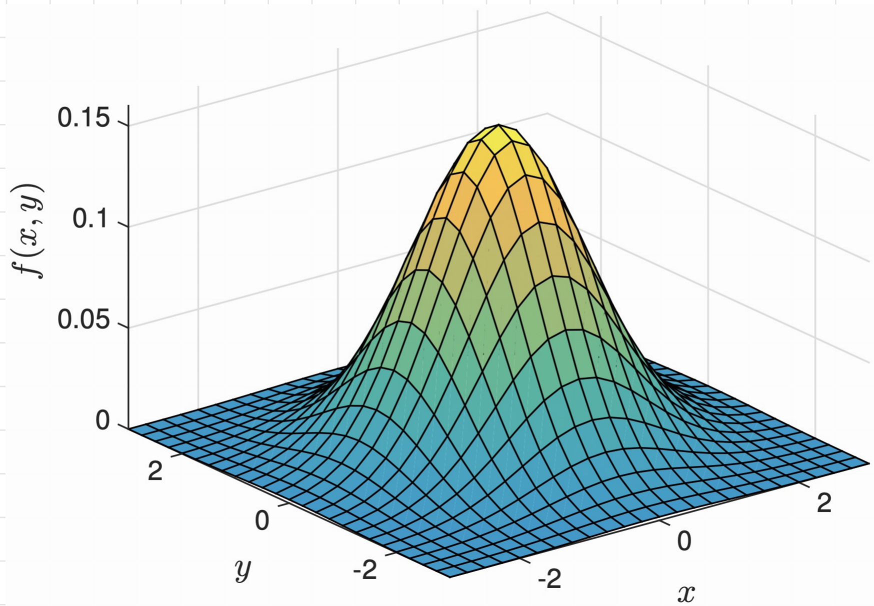
$$= a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) = a^2 + b^2$$

Not so obvious: $aX + bY$ is Normal

"Proof": Since X, Y are independent their joint p.d.f. is

$$f(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

This function is rotationally symmetric around 0



So $aX + bY = t$ is a vertical "slice" that can be rotated to $X = \dfrac{t}{\sqrt{a^2+b^2}}$ (preserves distance from 0)

Details: Note 21

Fact: If $X \sim N(0,1)$ and $Y \sim N(0,1)$ are independent, and a, b are constants, then

$$aX + bY \sim N(0, a^2 + b^2)$$

Generalization: If $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$ are independent, then

$$aX + bY \sim N(a\mu_x + b\mu_y,\ a^2\sigma_x^2 + b^2\sigma_y^2)$$

Proof: Apply above Fact to standard normals

$$\frac{X - \mu_x}{\sigma_x} \quad \text{and} \quad \frac{Y - \mu_y}{\sigma_y}$$

Central Limit Theorem

Recall: average of many independent samples of the same r.v.

$X_1, X_2, \ldots$ i.i.d. $\quad E[X_i] = \mu \quad Var(X_i) = \sigma^2$

Sample average: $\frac{1}{N} S_N$ where $S_N = X_1 + \cdots + X_N$

Amazing Fact: As $N \to \infty$, the distribution of $\frac{1}{N} S_N$ approaches $N(\mu, \sigma^2/N)$

$E[\frac{1}{N} S_N] = \frac{1}{N} \sum E[X_i] = \mu. \qquad Var(\frac{1}{N} S_N) = \frac{1}{N^2} \sum Var(X_i) = \sigma^2/N.$

Scale $\frac{1}{N}S_N$ so that limit is standard normal:

$$\frac{\frac{1}{N}S_N - \mu}{\sigma/\sqrt{N}} = \frac{S_N - N\mu}{\sigma\sqrt{N}}$$

Central Limit Theorem For i.i.d r.v.'s X_i with $E[X_i]=\mu$, $Var(X_i)=\sigma^2$, the distribution of $\frac{S_N-N\mu}{\sigma\sqrt{N}}$ converges to $N(0,1)$ as $N\to\infty$

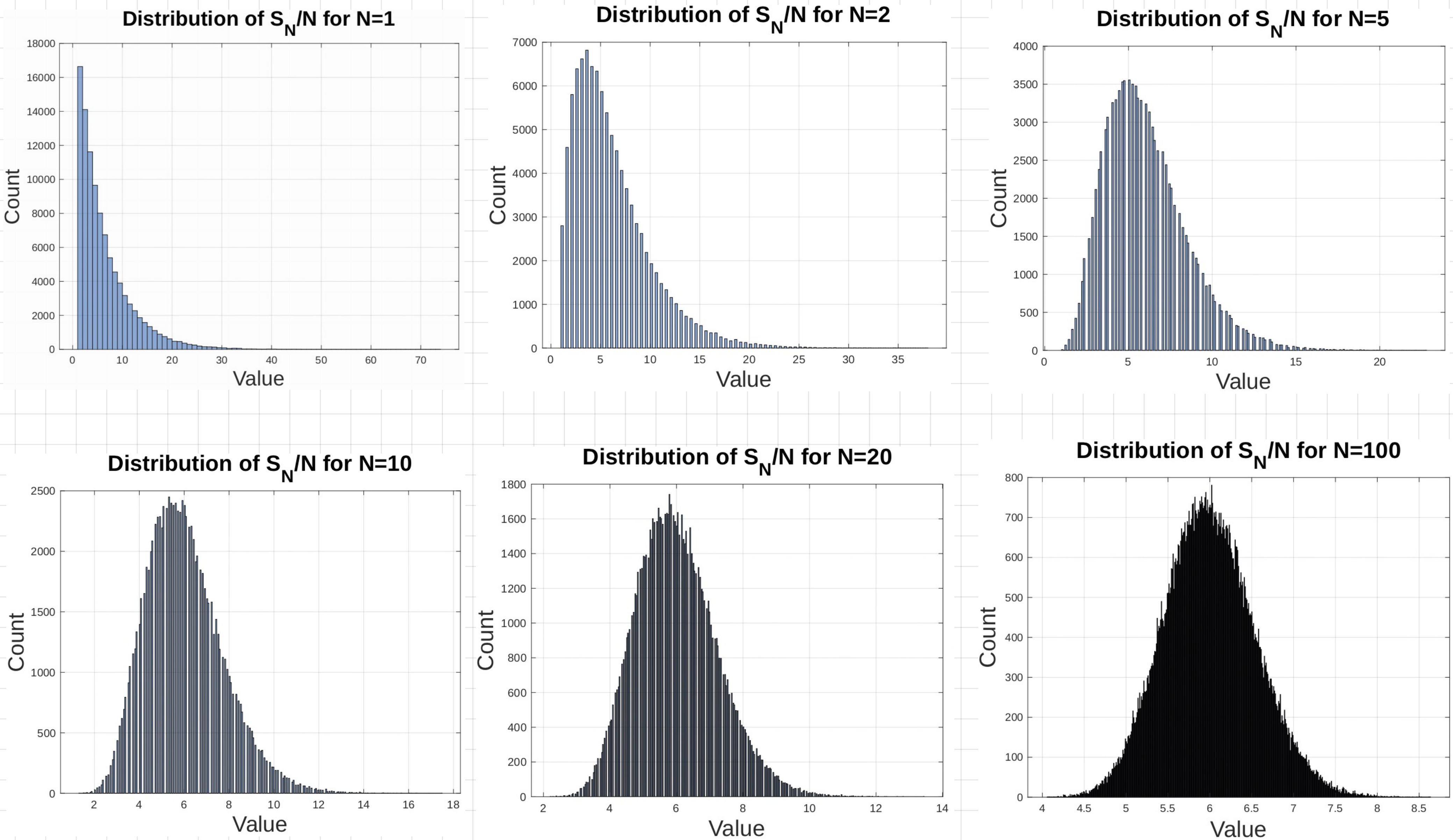
I.e.:
$$\Pr\left[\frac{S_N-N\mu}{\sigma\sqrt{N}} \leq c\right] \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-x^2/2}\,dx \quad \text{as } N\to\infty$$

for any constant c

CLT: Example

$X_i \sim \mathrm{Geom}(1/6)$. Distribns. of $\frac{1}{N}(X_1+\dots+X_N)$

$\mu = 1/p = 6 \qquad \sigma^2 = \frac{1-p}{p^2} = 30$



"width" of distribution $\approx \sigma/\sqrt{N} \longrightarrow 0$ as $N \to \infty$