CS70 - Spring 2024

Lecture 25 - April 18

Recap of Previous Lecture

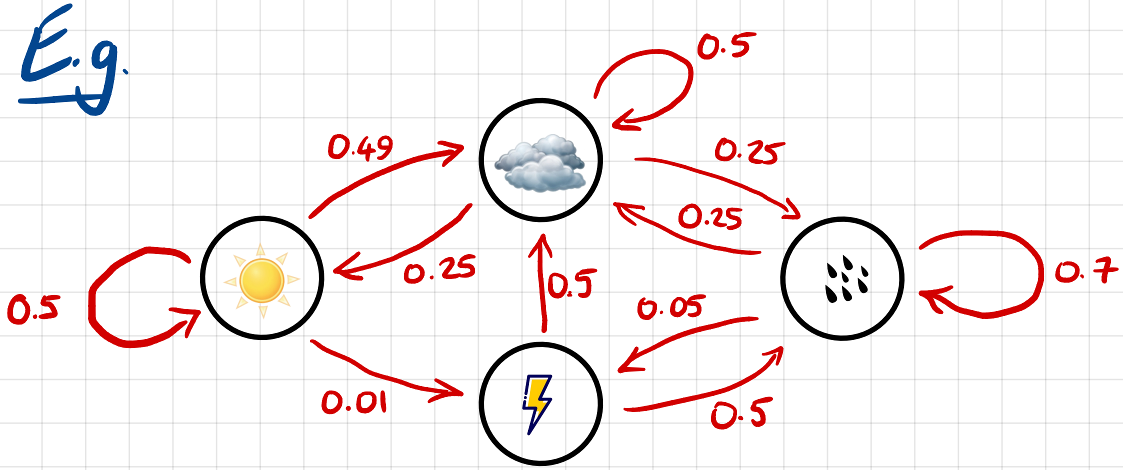
Markov chain:

- Set of states $\mathcal{K} = \{1, \dots, K\}$
- $K \times K$ transition matrix P s.t.

$$P(i,j) \geq 0 \quad \forall i,j$$

$$\sum_{j \in \mathcal{K}} P(i,j) = 1 \quad \forall i$$

E.g.



$$P = \begin{pmatrix} 0.5 & 0.25 & 0.25 & 0 \\ 0.49 & 0.5 & 0 & 0.01 \\ 0.25 & 0 & 0.7 & 0.05 \\ 0.5 & 0 & 0.5 & 0 \end{pmatrix}$$

Markov chains (cont.)

- **Time evolution :**

$X_0 \sim \pi_0$: initial state

$X_n \sim \pi_n$: state at time n

$Pr[X_n = j \mid X_{n-1} = i] = P(i,j)$ [independent of $X_0, X_1, \ldots X_{n-2}$]

$\Rightarrow \pi_n = \pi_{n-1} P = \pi_0 P^n$

$$\begin{bmatrix} - \pi_n - \end{bmatrix} = \begin{bmatrix} - \pi_{n-1} - \end{bmatrix} \begin{pmatrix} & & \\ & P & \\ & & \end{pmatrix}$$

Markov chains (cont.)

- **Invariant / Stationary Distribution:**

  A vector π satisfying $\boxed{\pi P = \pi}$

- Can compute π by solving the balance equations:

$$\boxed{\pi(j) = \sum_{i \in K} \pi(i)\, P(i,j)} \qquad i = 1, 2, \ldots, K$$

and normalizing so that $\sum_{j \in K} \pi(j) = 1$

- P is irreducible if $\exists$ path $i \rightsquigarrow j$ $\forall i j$
- P is aperiodic if the set of all path lengths $i \rightsquigarrow j$ has no non-trivial common factor

  [Can make any P aperiodic if necessary by adding a loop to every state]

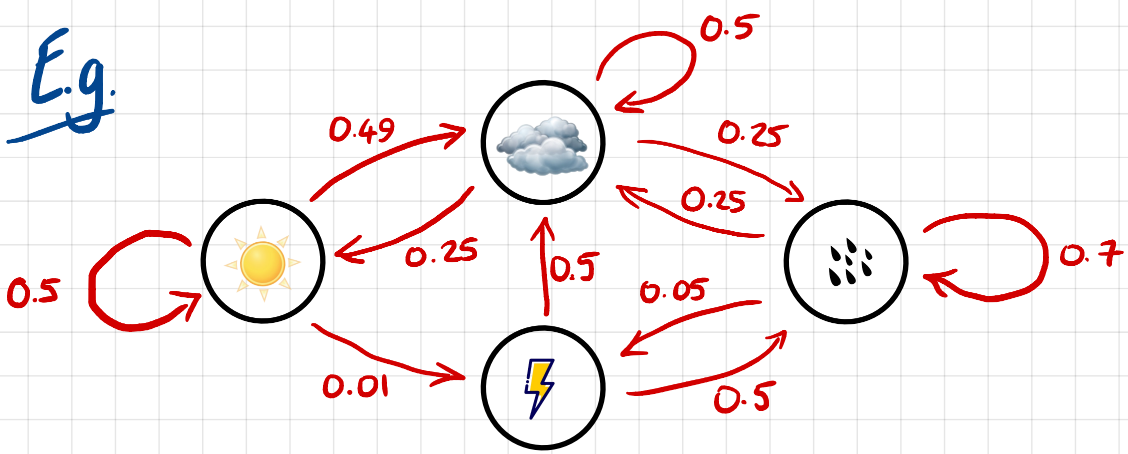
Markov chains (cont.)

Fundamental Theorem

If Markov chain P is irreducible & aperiodic then it has a unique invariant π, $\pi(i) > 0 \ \forall i$, and

$$\boxed{\Pr[X_n = i] \xrightarrow[n\to\infty]{} \pi(i) \quad \forall i \ \forall X_0}$$

E.g.



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$$\pi = \frac{1}{1358}[550,\ 275,\ 505,\ 28] \approx [0.405,\ 0.202,\ 0.372,\ 0.021]$$

$$\Pr[X_n = \text{☀️}] \longrightarrow \frac{505}{1358} \approx 0.202 \text{ as } n\to\infty$$

Today

- More examples
  - card shuffling
  - random walk on a graph
- Hitting time
- Gambler's Ruin

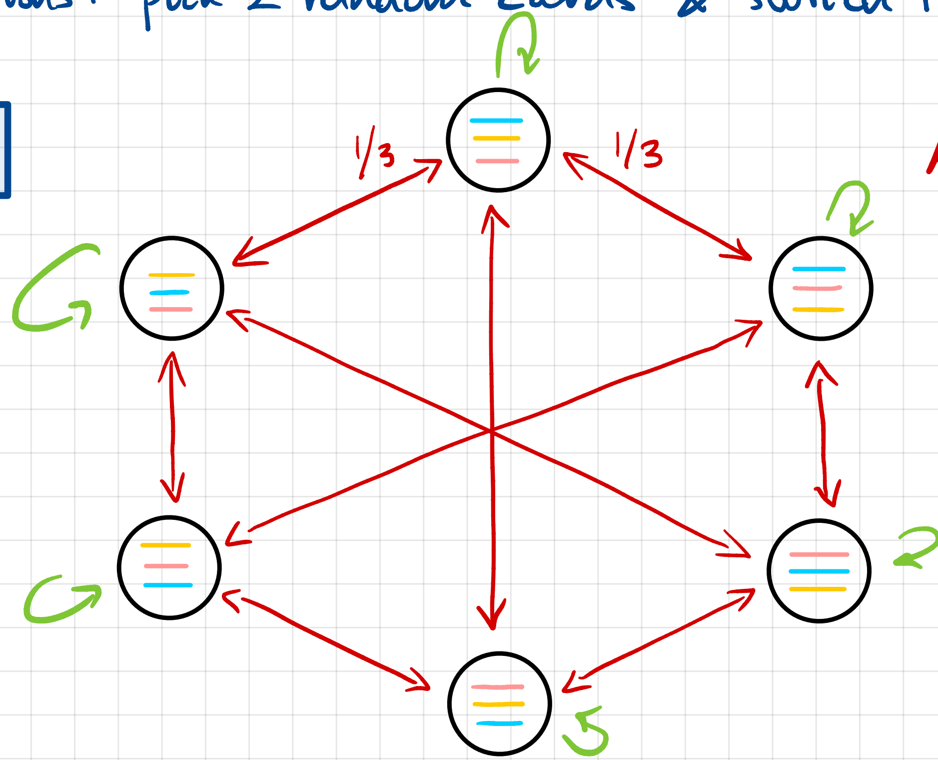
Examples

1. Recall the slow card shuffle:

States: all $N!$ permutations of the deck (N cards)

Transitions: pick 2 random cards & switch them

$\boxed{N=3}$



All trans. probs. $= \dfrac{2}{N(N-1)} = \dfrac{1}{3}$

$\dfrac{1}{\binom{N}{2}}$

This chain is <u>irreducible</u> ($\forall N$) because we can transform any permutation into any other using a seq. of transpositions

We can make it <u>aperiodic</u> by adding a small loop at every state

Irreducible & aperiodic $\Rightarrow$ $\exists$ unique invariant π

<u>Claim</u>: π is <u>uniform</u> over all $K = N!$ permutations

<u>Proof</u>: Follows from more general property that P is <u>symmetric</u>, i.e., $P(i,j) = P(j,i) \ \forall i,j$

Balance equations:

$$\pi(j) = \sum_{i=1}^{K} \pi(i) P(i,j) = \sum_{i=1}^{K} \pi(i) P(j,i)$$

symmetry of P

Plugging in $\pi(i) = \frac{1}{K} \ \forall i$:

$$\frac{1}{K} = \sum_{i=1}^{K} \frac{1}{K} P(j,i) = \frac{1}{K} \sum_{i=1}^{K} P(j,i) = \frac{1}{K} \checkmark$$

Hence π uniform is invariant.

<u>Corollary</u>: Starting from any ordering of the cards, if we repeatedly perform random transp's. we will converge to a uniformly random ordering!

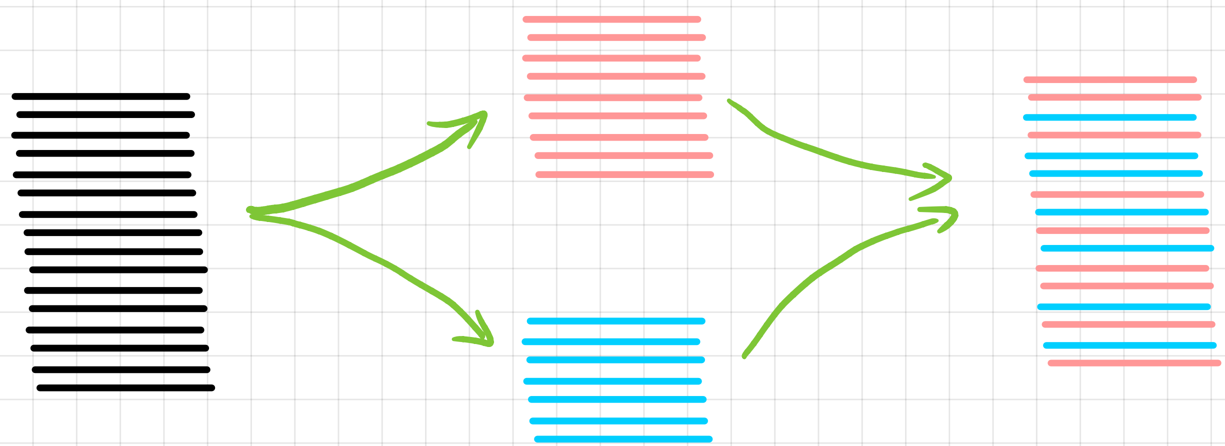
Q: How many transpositions until deck is close to uniform

A: $O(N \log N)$ (where $N = \#$ of cards)

[see CS 174]

Aside: Mathematical model of "real" riffle shuffle

- split deck into two parts using $\mathrm{Bin}(N, 1/2)$ distribution
- interleave the two parts by dropping next card from Left/Right hand w.prob. $\frac{L}{L+R}$ / $\frac{R}{L+R}$



Fact: $O(\log N)$ of these shuffles suffice

when $N=52$, 7 shuffles suffice [Bayer/Diaconis]

"Trailing the Dovetail Shuffle to Its Lair"

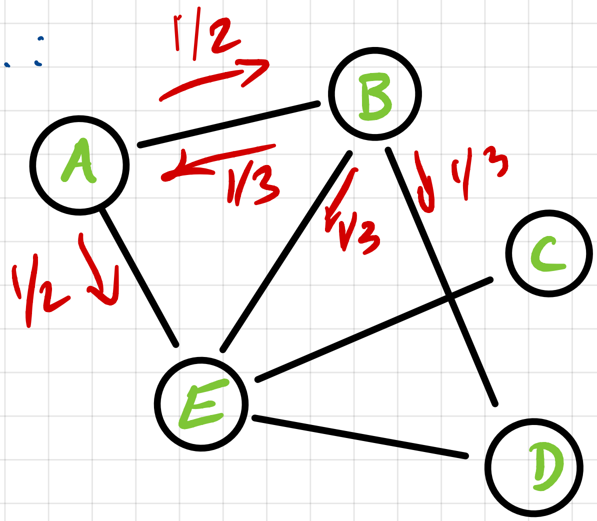
Random Walk on a Graph

Let $G = (V, E)$ be a connected undirected graph

Random walk on G is the Markov chain with states V that at each step moves to a random neighbor of the current vertex

i.e., $$P(u,v) = \begin{cases} \frac{1}{\deg(u)} & \text{if } \{u,v\} \in E \\ 0 & \text{otherwise} \end{cases}$$

E.g.:



$$P = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 1/2 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{pmatrix} \end{array}$$

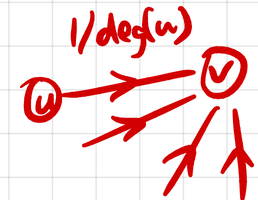
Assume G is connected and not bipartite

Then random walk converges to a unique invariant distribution π

$\boxed{Q:}$ What is π ?

$\boxed{A:}$ $$\pi(u) = \frac{\deg(u)}{2|E|}$$

normalizing factor: $\sum_{u \in V} \deg(u) = 2|E|$

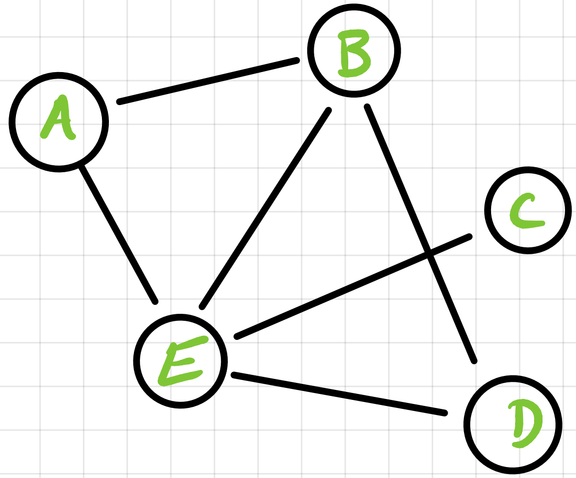


Proof : Check the balance equations :

$$\pi(v) = \sum_{u \in V} \pi(u) P(u,v) = \sum_{u:\{u,v\}\in E} \pi(u) \cdot \frac{1}{\deg(u)}$$

$$\frac{\deg(v)}{2|E|} = \sum_{u:\{u,v\}\in E} \frac{\deg(u)}{2|E|} \times \frac{1}{\deg(u)} = \frac{1}{2|E|} \sum_{u:\{u,v\}\in E} 1 = \frac{\deg(v)}{2|E|} \checkmark$$

Example:



$2|E| = 12$

$$\pi(A) = \frac{2}{12} = \frac{1}{6}$$

$$\pi(B) = \frac{3}{12} = \frac{1}{4}$$

$$\pi(C) = \frac{1}{12}$$

$$\pi(D) = \frac{2}{12} = \frac{1}{6}$$

$$\pi(E) = \frac{4}{12} = \frac{1}{3}$$

Hitting Time

* except that t may be absorbing

$\boxed{Q:}$ Let P be an irreducible* Markov chain

What is the expected no. of steps to reach state t starting from state i ?

Define $\beta(i) := E[\#\text{ steps to reach } t \text{ starting from } i]$

Then $\quad \beta(i) = 1 + \sum_{j \in K} P(i,j)\beta(j) \qquad \forall i \neq t$

$$\beta(t) = 0$$

"First step" equations — another linear system

Define $\beta(i) := E[\#\text{ steps to reach } t \text{ starting from } i]$

Then $\beta(i) = 1 + \sum_{j \in K} P(i,j)\,\beta(j) \qquad \forall i \neq t$

$\beta(t) = 0$

Recall: "law of total probability" $Pr[A] = \sum_{B_i} Pr[A \mid B_i]\, Pr[B_i]$

$\{B_i\}$ a partition

A bit more formally . . .

Prob. space: all paths ending at t

X_i = length of path assuming it starts at i

Then $E[X_i] = \sum_{j \in K} E[X_i \mid \text{1st step is } i \to j]\, Pr[\text{1st step is } i \to j]$

"law of total expectation"

$= \sum_{j \in K} P(i,j)\, \underbrace{E[X_i \mid \text{1st step is } i \to j]}_{= 1 + E[X_j]}$

$= 1 + \sum_{j \in K} P(i,j)\,\beta(j)$

E.g. :

For $i \in \{A, B\}$ let $\beta(i) = E[\#\text{ steps to reach } B \text{ starting from } i]$

Then
$$\beta(A) = 1 + (1-p)\beta(A) + p\beta(B)$$
$$\beta(B) = 0$$

Solve : $\beta(A) = 1 + (1-p)\beta(A)$

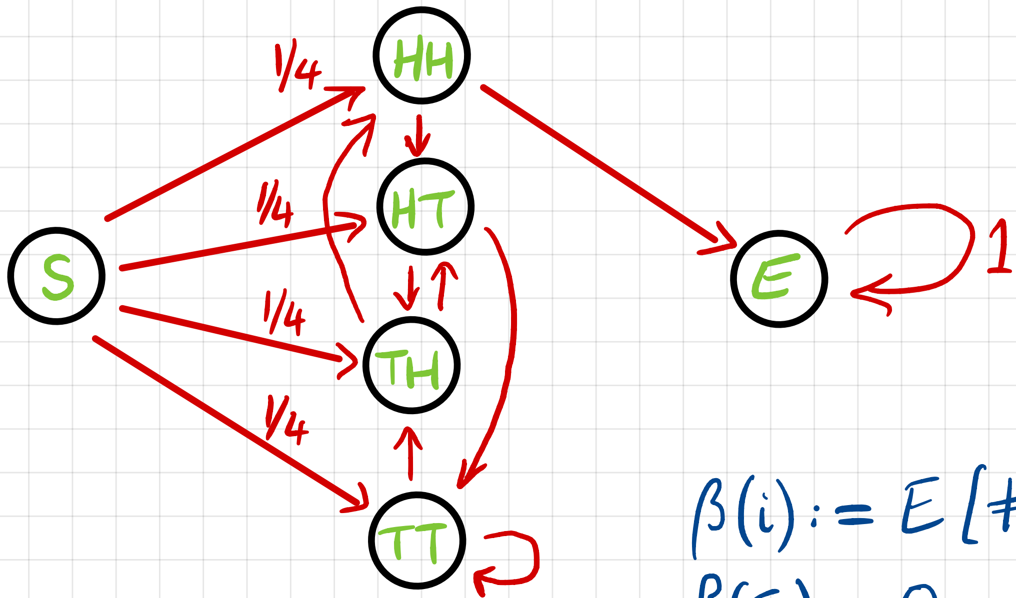
$$\Rightarrow \boxed{\beta(A) = 1/p}$$

<u>Note</u> : Alternative proof of $E[X] = 1/p$ for $X \sim \text{Geom}(p)$

Extension: toss a fair coin until you get HHH

$X = \#$ tosses

What is $E[X]$?



All remaining trans. probs. are 1/2

$\beta(i) := E[\#\ \cancel{\text{steps}}^{\text{tosses}}$ to reach E starting at $i]$

$\beta(E) = 0$

$\beta(S) = \boxed{2} + \frac{1}{4}\Big(\beta(HH) + \beta(HT) + \beta(TH) + \beta(TT)\Big)$

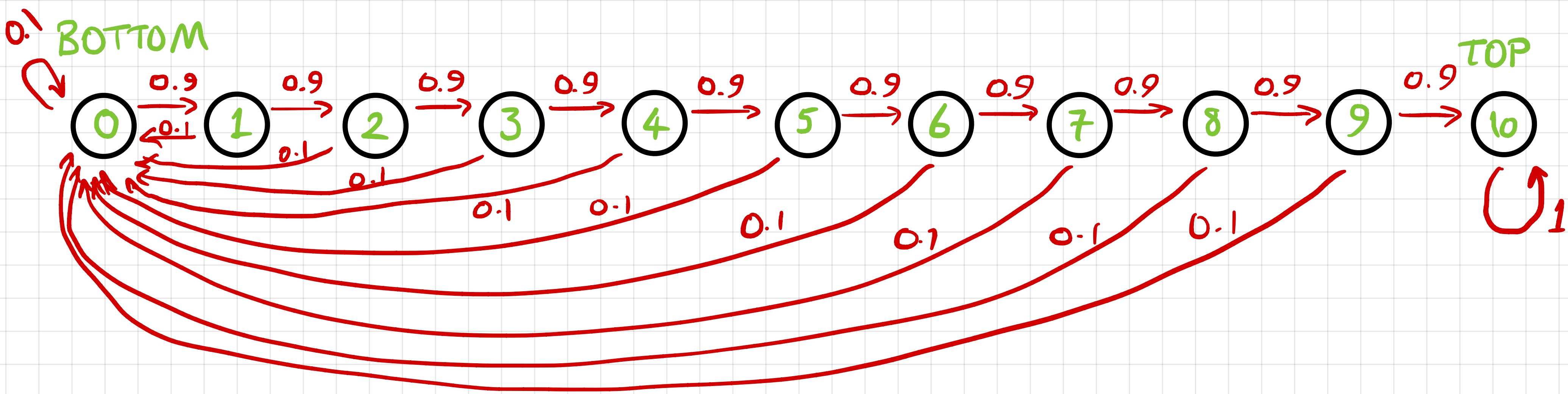
$\beta(HH) = 1 + \frac{1}{2}\beta(E) + \frac{1}{2}\beta(HT)$

$\beta(HT) = 1 + \frac{1}{2}\beta(HH) + \frac{1}{2}\beta(TH)$

$\vdots$

Solve: $\boxed{\beta(S) = 14}$

Example: Climbing a (very slippery) 10-rung ladder
On each step, slip down to bottom w. prob. 0.1



$\boxed{Q:}$ What is expected time to reach Top from Bottom?

Let $\beta(i) := E[\#\text{steps to reach Top starting at } i]$ $i = 0, 1, \ldots, 10$

First step equns:

$$\beta(i) = 1 + (1-p)\beta(0) + p\beta(i+1)$$

$$\beta(10) = 0$$

$$\boxed{p = P(i, i+1) = 0.9}$$

First step equns:

$$\beta(i) = 1 + (1-p)\beta(0) + p\beta(i+1)$$

$$\beta(10) = 0$$

$p = P(i, i+1) = 0.9$

Solve . . .

$$\beta(9) = 1 + (1-p)\beta(0) + p\beta(10) = 1 + (1-p)\beta(0)$$

$$\beta(8) = 1 + (1-p)\beta(0) + p\left[1 + (1-p)\beta(0)\right] = (1+p)\left[1 + (1-p)\beta(0)\right]$$

$$\beta(7) = 1 + (1-p)\beta(0) + p(1+p)\left[1 + (1-p)\beta(0)\right] = (1+p+p^2)\left[1 + (1-p)\beta(0)\right]$$

$$\vdots$$

$$\beta(i) = (1 + p + \cdots + p^{9-i})\left[1 + (1-p)\beta(0)\right] \qquad i = 0, 1, \ldots, 9$$

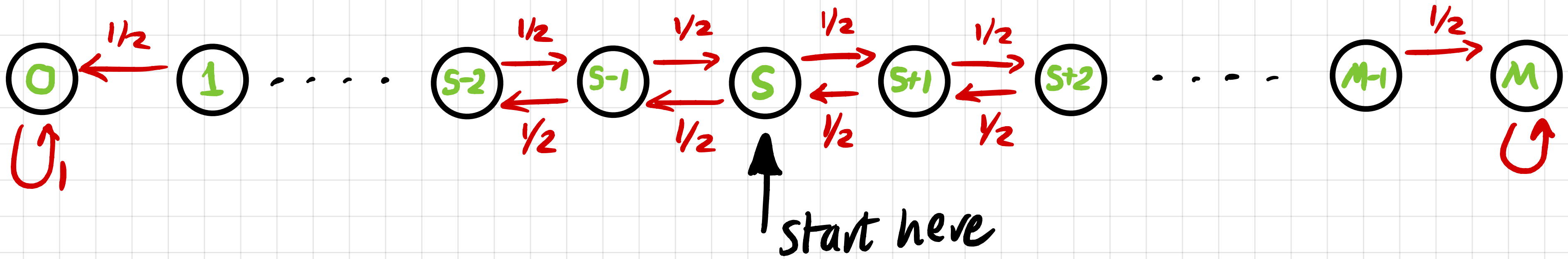
$$= \frac{1 - p^{10-i}}{1-p}\left[1 + (1-p)\beta(0)\right]$$

$$\Rightarrow \boxed{\beta(0) = \frac{p^{-10} - 1}{1-p}}$$

For $p = 0.9$, $\beta(0) \approx 18.7$

Gambler's Ruin

Recall: Fair game: win/lose \$1 each with prob. $1/2$
Start with \$S, end when reach \$0 or \$M



Q: What is the probability we hit 0 before hitting M?

Define $\alpha(i) := \Pr[\text{hit } 0 \text{ before } M \text{ starting at } i]$

Then $\alpha(M) = 0 \qquad \alpha(0) = 1$

$$\alpha(i) = \frac{1}{2}\alpha(i+1) + \frac{1}{2}\alpha(i-1)$$

first step equations

General MC: $\alpha(i) = \sum_j P(i,j)\,\alpha(j)$ [law of total prob.] $\quad i \neq 0, M$

Define $\alpha(i) := \Pr[\text{hit } 0 \text{ before } M \text{ starting at } i]$

$$\left.\begin{aligned} &\text{Then} \quad \alpha(M) = 0 \qquad \alpha(0) = 1 \\ &\alpha(i) = \tfrac{1}{2}\alpha(i+1) + \tfrac{1}{2}\alpha(i-1) \end{aligned}\right\} \text{first step equations}$$

$$\alpha(M-1) = \tfrac{1}{2}\alpha(M) + \tfrac{1}{2}\alpha(M-2) \Rightarrow \alpha(M-1) = \tfrac{1}{2}\alpha(M-2)$$

$$\alpha(M-2) = \tfrac{1}{2}\alpha(M-1) + \tfrac{1}{2}\alpha(M-3) \Rightarrow \alpha(M-2) = \tfrac{2}{3}\alpha(M-3)$$

$$\alpha(M-3) = \tfrac{1}{2}\alpha(M-2) + \tfrac{1}{2}\alpha(M-4) \Rightarrow \alpha(M-3) = \tfrac{3}{4}\alpha(M-4)$$

$$\vdots$$

$$\alpha(M-j) = \frac{j}{j+1}\alpha(M-j-1) = \frac{j}{j+1}\cdot\frac{j+1}{j+2}\cdot\frac{j+2}{j+3}\cdots\frac{k-1}{k}\,\alpha(M-k)$$

$$= \frac{j}{k}\,\alpha(M-k)$$

Set $k = M \Rightarrow \alpha(M-j) = \frac{j}{M}\alpha(0) = \frac{j}{M}$

Set $j = M-S \Rightarrow \boxed{\alpha(S) = \frac{M-S}{M}}$ $\longleftarrow$ $\Pr[\text{hit } 0 \text{ before } M] \propto$ distance of start point S from M