CS70: Lecture 3. Induction!

1. The natural numbers.
2. Seven year old Gauss.
3. ...and Induction.
4. Simple Proof.
5. Two coloring map

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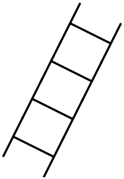
(mostly) Next time:

1. Strengthening induction.
2. Tiling Cory Hall courtyard.
3. Horses with one color...

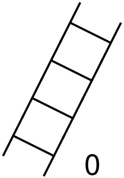
The naturals.

The naturals.



The naturals.

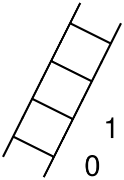
0,



0

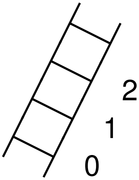
The naturals.

0, 1,



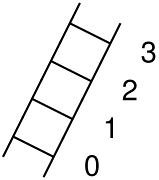
The naturals.

0, 1, 2,



The naturals.

0, 1, 2, 3,



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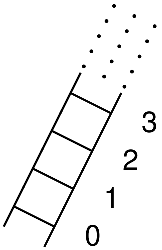
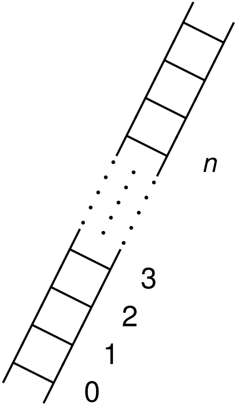


0, 1, 2, 3,

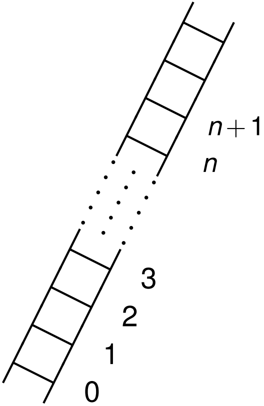
. . . ,

The naturals.



0, 1, 2, 3,
…, n,

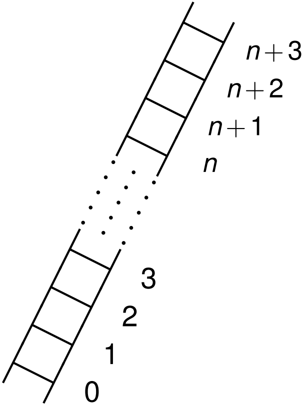
The naturals.



$0, 1, 2, 3,$

$\ldots, n, n+1,$

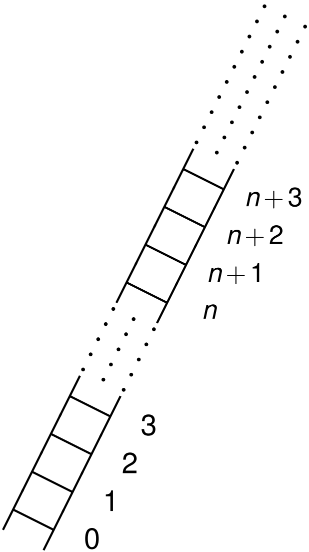
The naturals.



0, 1, 2, 3,

$\ldots, n, n+1, n+2, n+3,$

The naturals.



0, 1, 2, 3,

$\ldots, n, n+1, n+2, n+3, \ldots$

A Story about a 7-year old Gauss.

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Teacher: Hello class.

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A Story about a 7-year old Gauss.

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Gauss: It's 5050!

A Story about a 7-year old Gauss.

Teacher: Hello class.

Teacher: Please add the numbers from 1 to 100.

Gauss: It's 5050! (that is, $50 \times 101 = \frac{(100)(101)}{2}$)

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=1}^{n} i = \frac{n(n+1)}{2})$

Gauss and Induction

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Idea: assume predicate for $n = k$.

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Is predicate true for $n = k + 1$?

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$\sum_{i=1}^{k+1} i$

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$\sum_{i=1}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1)$

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How about $k+2$.

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How about $k + 2$. Same argument starting at $k + 1$ works!

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Is this a proof? It shows that we can always move to the next step.

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Need to start somewhere.

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Need to start somewhere. $\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}$

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Statement is true for $n = 0$

Gauss and Induction

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Statement is true for $n = 0$

plus inductive step

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Induction Step.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}$ **Base Case.**

Statement is true for $n = 0$
plus inductive step $\Longrightarrow$ true for $n = 1$

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Statement is true for $n = 0$
plus inductive step $\implies$ true for $n = 1$
plus inductive step $\implies$ true for $n = 2$

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...
true for $n = k$

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true for $n = k \implies$ true for $n = k+1$

$\ldots$

Predicate True for all natural numbers!

Proof by Induction.

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

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- For all natural numbers n, $1+2\cdots n = \frac{n(n+1)}{2}$.

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1+2\cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

Induction

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The basic form

Induction

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The basic form

- Prove $P(0)$. “Base Case”.

Induction

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The basic form

- Prove $P(0)$. "Base Case".
- $P(k) \implies P(k+1)$
  - Assume $P(k)$, "Induction Hypothesis"

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
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Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
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- The sum of the first n odd integers is a perfect square.

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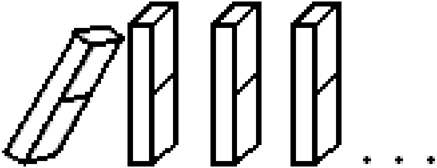
Notes visualization

An visualization: an infinite sequence of dominos.



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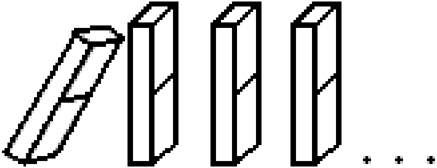

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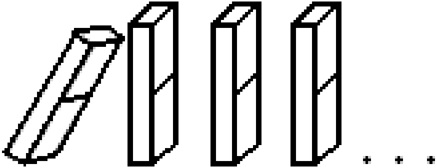

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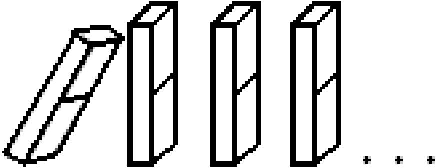

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Climb an infinite ladder?

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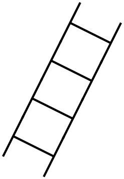

Climb an infinite ladder?

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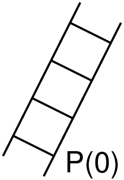

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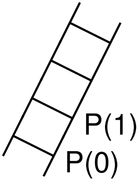

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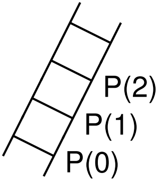

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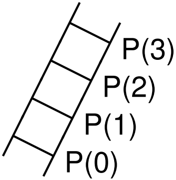

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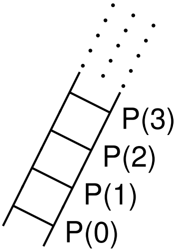

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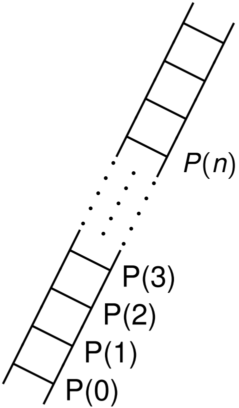

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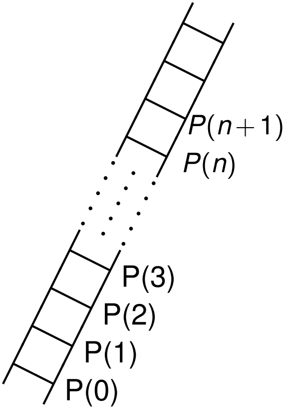

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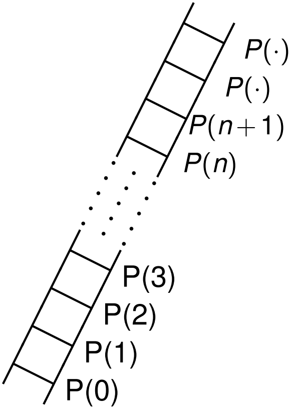

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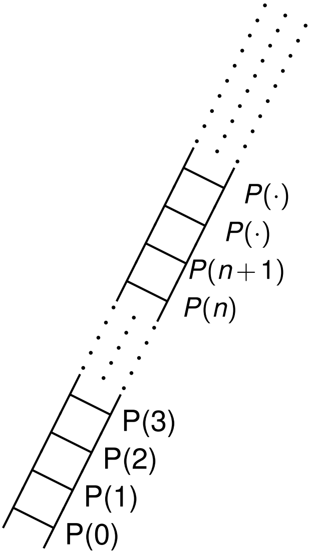

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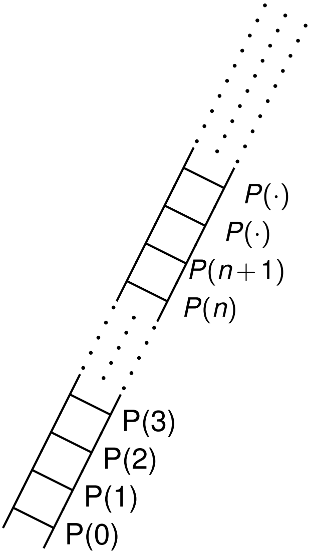

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Your favorite example of "forever"...

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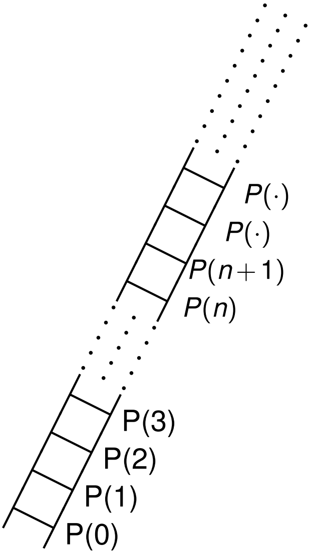

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Your favorite example of “forever”...or the integers...

Simple induction proof.

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$P(n+1)$!

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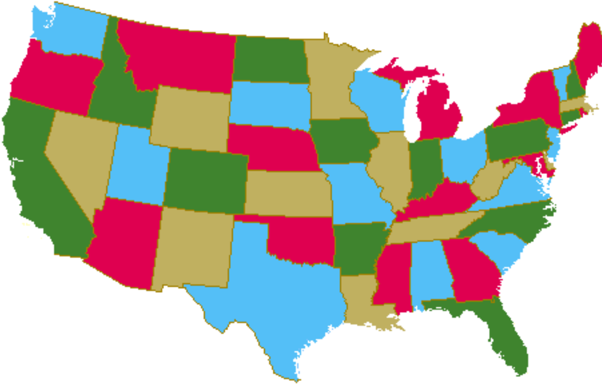
$(\forall n \in N)\,(P(n) \implies P(n+1))$. $\square$

Four Color Theorem.

Theorem: Any map can be colored so that those regions that share an edge have different colors.



Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

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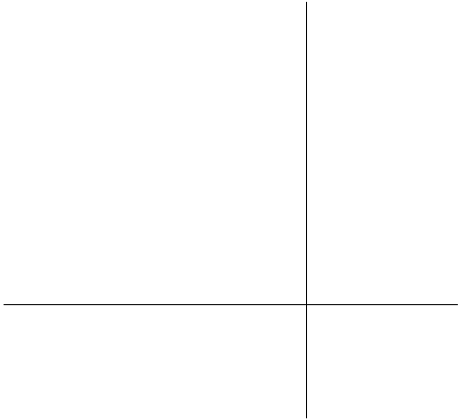

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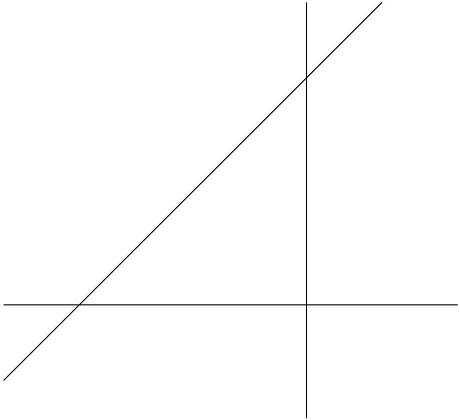

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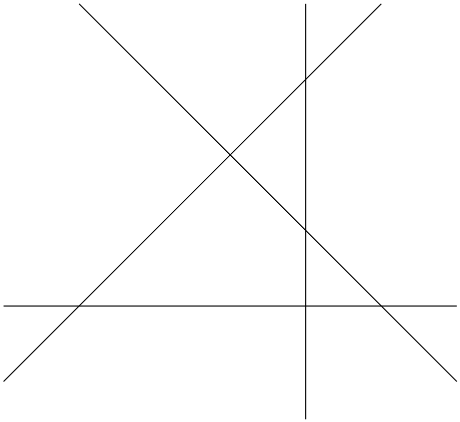

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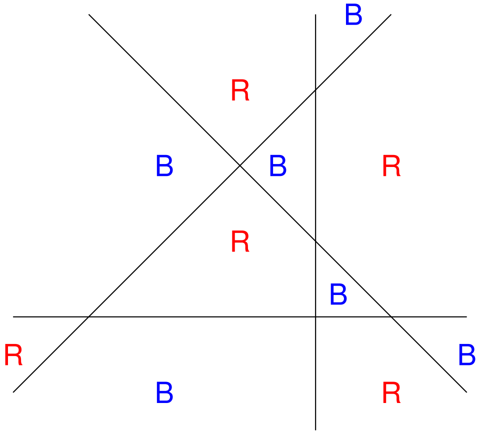

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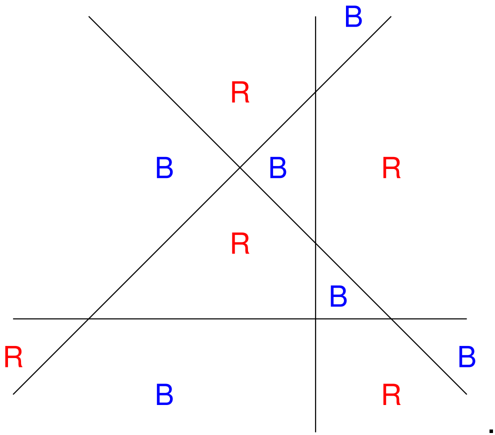

Fact: Swapping red and blue gives another valid coloring.

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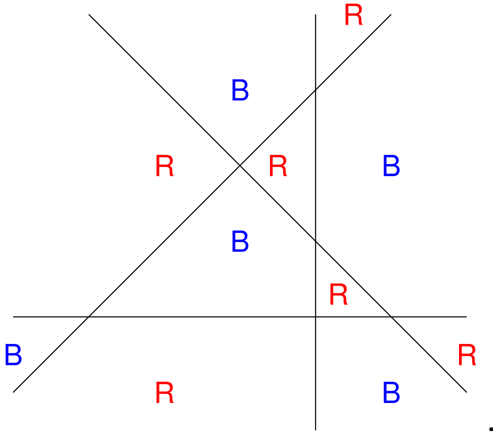

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Two color theorem: proof illustration.

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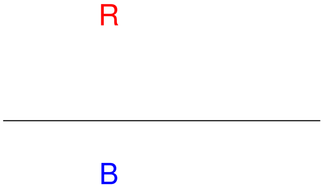

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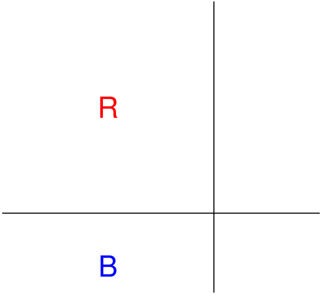
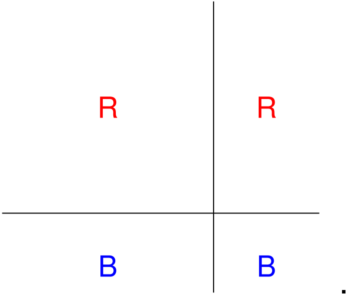


1. Add line.

Two color theorem: proof illustration.



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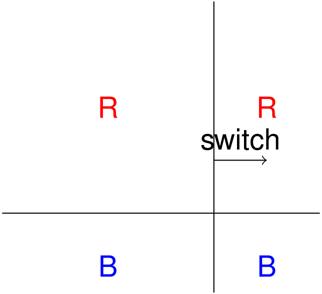

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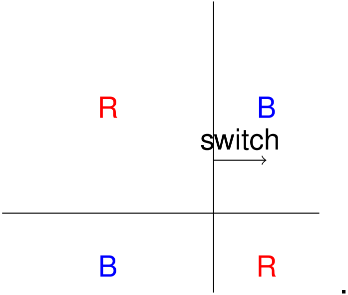

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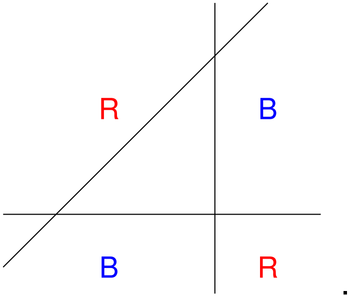

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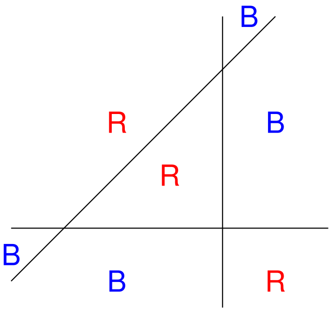

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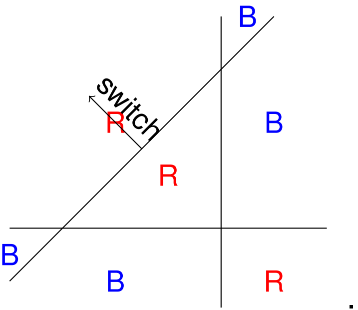

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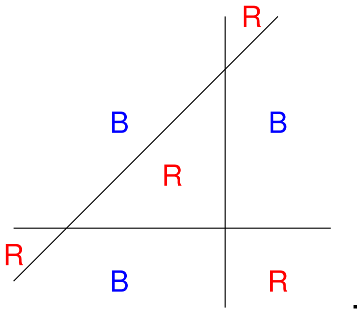

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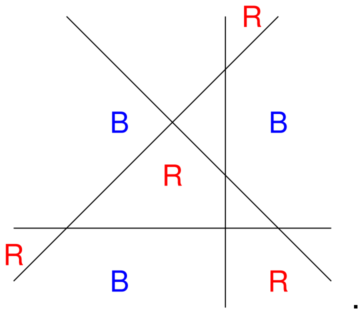

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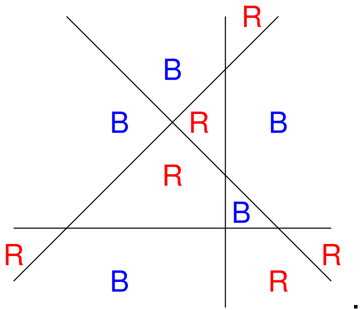

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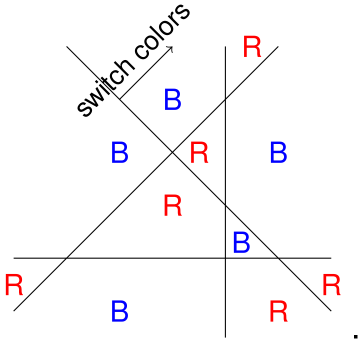

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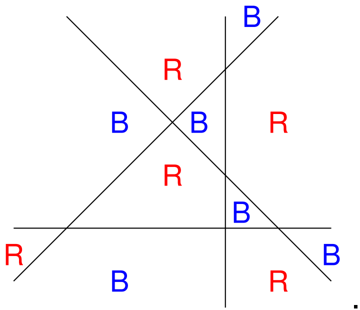

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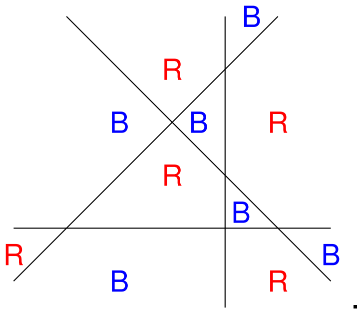

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Algorithm gives $P(k) \implies P(k+1)$.

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$(P(0)$

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Variations:

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Summary: principle of induction.

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Statement to prove: $P(n)$ for n starting from n_0

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Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!