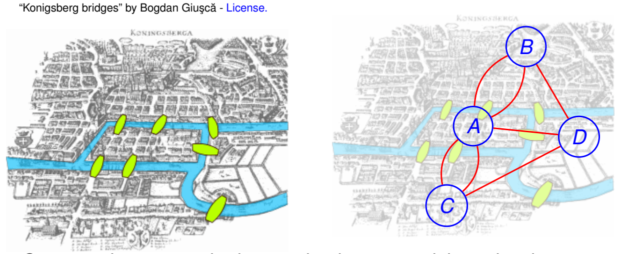
Introduction to Graphs

Graphs!
- Euler
- Definitions: model.
- Euler Again!!

Konigsberg bridges problem.

Can you make a tour visiting each bridge exactly once?

“Konigsberg bridges” by Bogdan Giuşcă - License.



Can you draw a tour in the graph where you visit each edge once? Yes? No?
We will see!

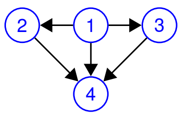
Graphs: formally.

Graph: $G = (V, E)$.
- V - set of vertices.
  - $\{A, B, C, D\}$
- $E \subseteq V \times V$ - set of edges.
  - $\{\{A,B\},\{A,B\},\{A,C\},\{A,C\},\{B,D\},\{A,D\},\{C,D\}\}$.

For CS 70, usually simple graphs.
- No parallel edges.

Multigraph above.

Directed Graphs



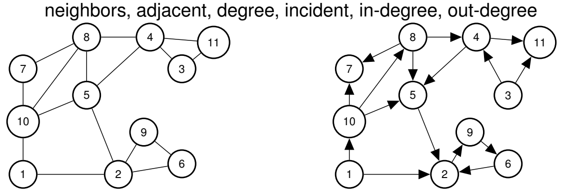
$G = (V, E)$.
- V - set of vertices.
  - $\{1, 2, 3, 4\}$
- E ordered pairs of vertices.
  - $\{(1,2),(1,3),(1,4),(2,4),(3,4)\}$

One way streets.
Tournament: 1 beats 2, ...
Precedence: 1 is before 2, ..

Social Network: Directed? Undirected?
- Friends. Undirected.
- Likes. Directed.

Graph Concepts and Definitions.

Graph: $G = (V, E)$
- neighbors, adjacent, degree, incident, in-degree, out-degree



Neighbors of 10? 1,5,7, 8.
- u is neighbor of v if $(u,v) \in E$ (or if $(v,u) \in E$).

Edge $(10,5)$ is incident to vertex 10 and vertex 5.
- Edge (u,v) is incident to u and v.

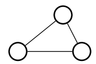
Degree of vertex 1? 2
- Degree of vertex u is number of incident edges.
  - Equals number of neighbors in simple graph.

Directed graph?
- In-degree of 10? 1 Out-degree of 10? 3

Quick Proof.

The sum of the vertex degrees is equal to
- (A) the total number of vertices, $|V|$.
- (B) the total number of edges, $|E|$.
- (C) Something else?

Not (A)! Triangle.



Not (B)! Triangle.

What could it be? For triangle number of edges is 3, the sum of degrees is 6.

Could it always be...$2|E|$?
- How many incidences does each edge contribute? 2.
  - $2|E|$ incidences are contributed in total!
- What is degree v? incidences contributed to v!
  - sum of degrees is total incidences ... or $2|E|$.

Thm: Sum of vertex degrees is $2|E|$.

Proof of “handshake” lemma.

Lemma: The sum of degrees is $2|E|$, for a graph $G=(V,E)$.

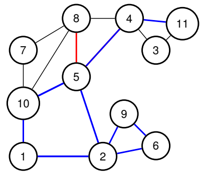
The number of edge-vertex incidences for an edge e is 2.

The total number of edge-vertex incidences is $2|E|$.

The sum of degrees is $2|E|$.

Handshake lemma: sum of number of handshakes of each person is twice the number of handshakes.

Paths, walks, cycles, tour.



A path in a graph is a sequence of edges.

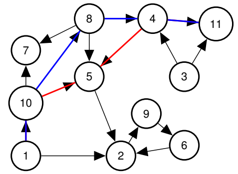
Path? $\{1,10\}$, $\{8,5\}$, $\{4,5\}$? No!
Path? $\{1,10\}$, $\{10,5\}$, $\{5,4\}$, $\{4,11\}$? Yes!

Path: $(v_1,v_2),(v_2,v_3),\ldots(v_{k-1},v_k)$.
Quick Check! Length of path? k vertices or $k-1$ edges.

Cycle: Path with $v_1=v_k$. Length of cycle? $k-1$ vertices and edges!
Path is usually *simple*. No repeated vertex!
Walk is sequence of edges with possible repeated vertex or edge.
Tour is walk that starts and ends at the same node.

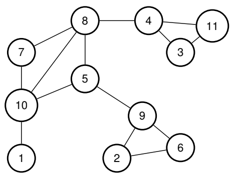
Quick Check!
Path is to Walk as Cycle is to ?? Tour!

Directed Paths.



Path: $(v_1,v_2),(v_2,v_3),\ldots(v_{k-1},v_k)$.
Paths, walks, cycles, tours ... are analogous to undirected now.

Connectivity



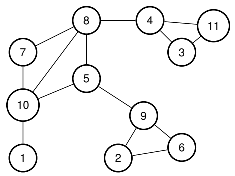
u and v are connected if there is a path between u and v.

A connected graph is a graph where all pairs of vertices are connected.

If one vertex x is connected to every other vertex.
Is graph connected? Yes? No?

Proof: Use path from u to x and then from x to v. □

May not be simple!
Either modify definition to walk.
Or cut out cycles.



Is graph above connected? Yes!

How about now? No!

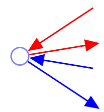
Connected Components? $\{1\},\{10,7,5,8,4,3,11\},\{2,9,6\}$.
A connected component is a maximal set of connected nodes in a graph.
Quick Check: Is $\{10,7,5\}$ a connected component? No.

Finally..back to Euler!

An Eulerian Tour is a tour that visits each edge exactly once.

Theorem: Any undirected graph has an Eulerian tour if and only if all vertices have even degree and is connected.
Proof of only if: Eulerian $\Longrightarrow$ connected and all even degree.

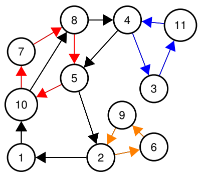
Eulerian Tour is connected so graph is connected.
Tour enters and leaves vertex v on each visit.
Uses two incident edges per visit. Tour uses all incident edges.
Therefore v has even degree. □



When you enter, you leave.
For starting node, tour leaves first ....then enters at end.

Finding a tour!

Proof of if: Even + connected $\Longrightarrow$ Eulerian Tour.
We will give an algorithm. First by picture.



1. Take a walk starting from v (1)
... till you get back to v.
2. Remove tour, C.
3. Let $G_1, \ldots, G_k$ be connected components.
   Each is touched by C.
   Why? G was connected.
   Let v_i be (first) node in G_i touched by C.
   Example: $v_1 = 1$, $v_2 = 10$, $v_3 = 4$, $v_4 = 2$.
4. Recurse on $G_1, \ldots, G_k$ starting from v_i
5. Splice together.
   1,10,7,8,5,10 ,8,4,3,11,4 5,2,6,9,2 and to 1!

General case: Recursive algorithm, proof by induction.

1. Take a walk from arbitrary node v, until you get back to v.

Claim: Do get back to v!
Proof of Claim: Even degree. If enter, can leave except for v. □

2. Remove cycle, C, from G.

Resulting graph may be disconnected. (Removed edges!)

Let components be $G_1, \ldots, G_k$.

Let v_i be first vertex of C that is in G_i.
Why is there a v_i in C?
G was connected $\Longrightarrow$
a vertex in G_i must be incident to a removed edge in C.

Claim: Each vertex in each G_i has even degree and is connected.
Prf: Tour C has even incidences to any vertex v. □

3. Find tour T_i of G_i starting/ending at v_i.
4. Splice T_i into C where v_i first appears in C.

Visits every edge once:
Visits edges in C exactly once.
By induction for all other edges by induction on G_i. □

Summary

Graphs.
Basics.
Connectivity.
Algorithm for Eulerian Tour.