More on Graphs

Types of graphs.

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- Complete Graphs.
- Trees.
- Planar Graphs.
- Hypercubes.

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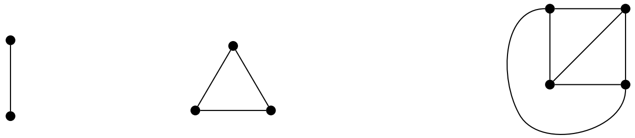
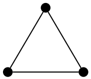
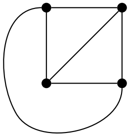
Complete Graph.



K_n complete graph on n vertices.

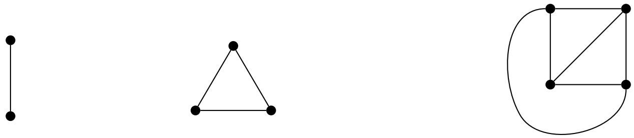
Complete Graph.





K_n complete graph on n vertices.
All edges are present.

Complete Graph.

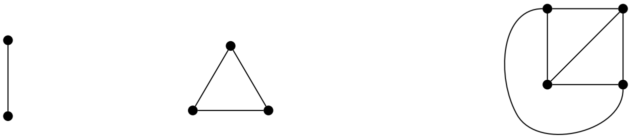


K_n complete graph on n vertices.

All edges are present.

Each vertex is adjacent to every other vertex.

Complete Graph.



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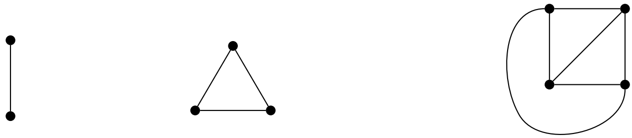

K_n complete graph on n vertices.

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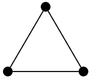
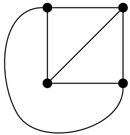
Each vertex is adjacent to every other vertex.

How many edges?

Complete Graph.





K_n complete graph on n vertices.

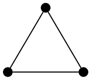
All edges are present.

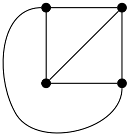
Each vertex is adjacent to every other vertex.

How many edges?

Each vertex is incident to $n-1$ edges.

Complete Graph.





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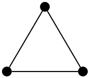
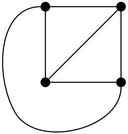
Each vertex is incident to $n-1$ edges.

Sum of degrees is $n(n-1)$.

Complete Graph.





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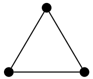
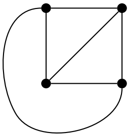
Sum of degrees is $n(n-1)$.

$\implies$ Number of edges is $n(n-1)/2$.

Complete Graph.

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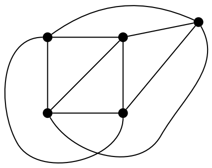
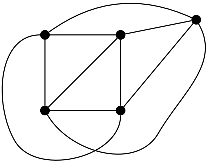
Remember sum of degree is $2|E|$.

K_4 and K_5



K_5 is not planar.

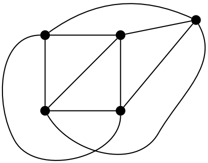
K_4 and K_5



K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

K_4 and K_5



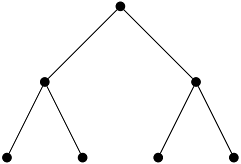
K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

We will prove this later!

Trees!

Graph $G = (V, E)$.
Binary Tree!



More generally.

Trees: Definitions

Definitions:

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Definitions:

A connected graph without a cycle.

Trees: Definitions

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- A connected graph without a cycle.
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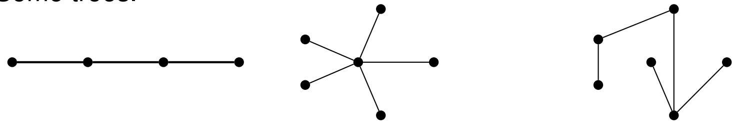
Some trees.



no cycle and connected?

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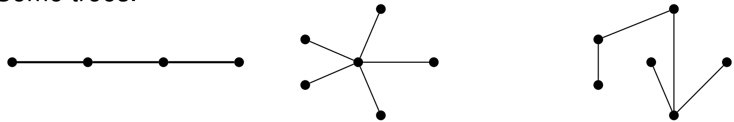

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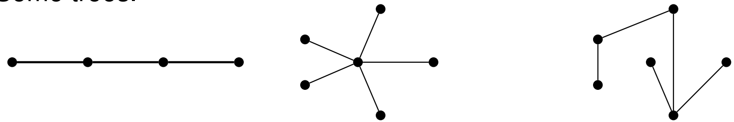

no cycle and connected? Yes.
$|V|-1$ edges and connected?

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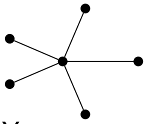
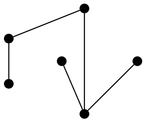

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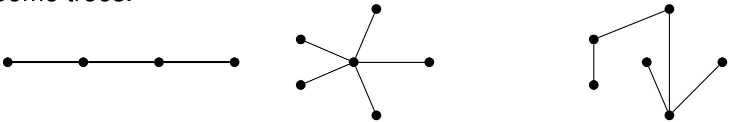

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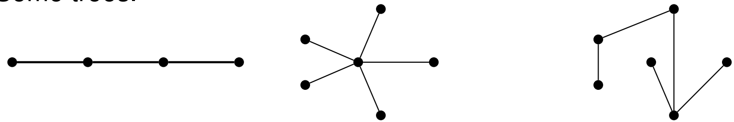

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removing any edge disconnects it. Harder to check.

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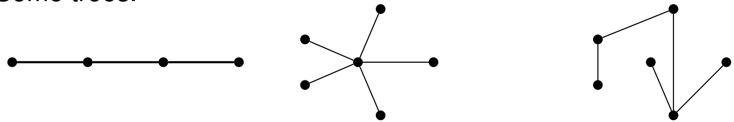

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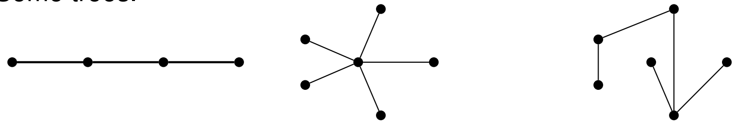

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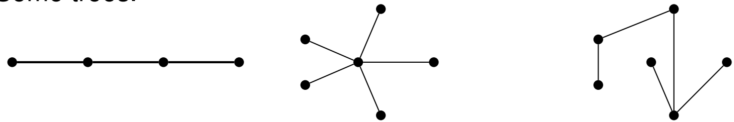

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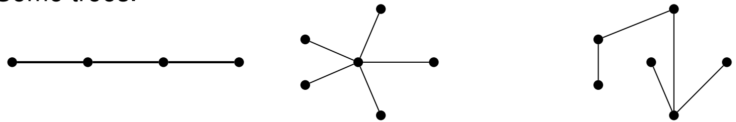

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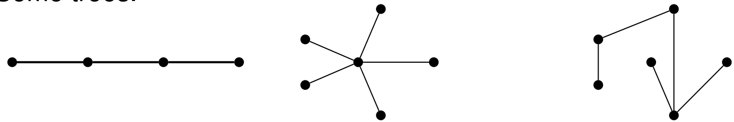

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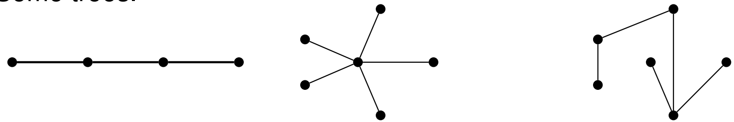

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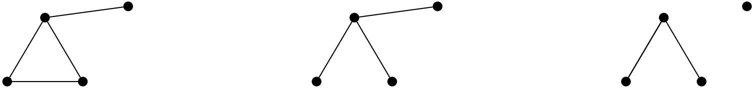
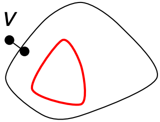
Tree or not tree!



Equivalence of Definitions

Thm:
"G connected and has $|V| - 1$ edges" $\equiv$
"G is connected and has no cycles."



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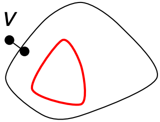

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Proof of $\Longrightarrow$ (only if):

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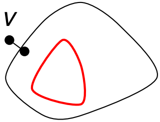

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Proof of $\implies$ (only if): By induction on $|V|$.

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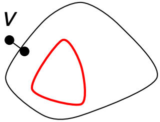

Thm:
"G connected and has $|V|-1$ edges" $\equiv$
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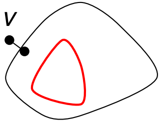
Proof of $\implies$ (only if): By induction on $|V|$.
Base Case: $|V|=1$. $0=|V|-1$ edges and has no cycles.

Equivalence of Definitions



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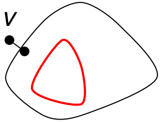

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Induction Step: Assume for G with up to k vertices. Prove for $k + 1$

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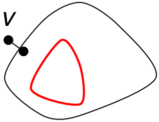

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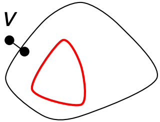

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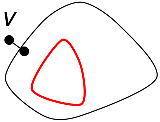
Might it be connected by just 1 edge?

Equivalence of Definitions



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Is there a Degree 1 vertex?

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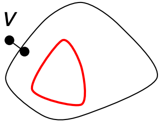

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Consider some vertex v in G. How is it connected to the rest of G?
Might it be connected by just 1 edge?
Is there a Degree 1 vertex?
Is the rest of G connected?

Equivalence of Definitions: Useful Lemma

Theorem:
"G connected and has $|V|-1$ edges" $\equiv$
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Lemma: If v is a degree 1 in connected graph G, $G-v$ is connected.
Proof:
For $x \neq v, y \neq v \in V$,

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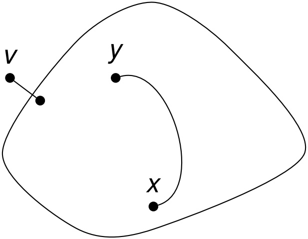

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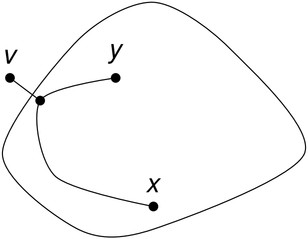
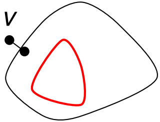
Lemma: If v is a degree 1 in connected graph G, $G-v$ is connected.
Proof:
For $x \neq v, y \neq v \in V$,
there is path between x and y in G since connected.
and does not use v (degree 1)
$\implies$ $G-v$ is connected. $\square$



Proof of only if.



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Proof of $\implies$: By induction on $|V|$.
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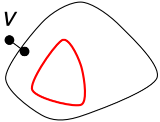

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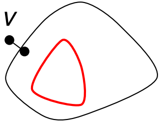

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Proof: First, connected $\implies$ every vertex degree ≥ 1.

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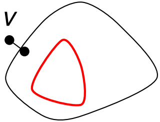

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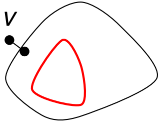
Sum of degrees is $2|V|-2$

Proof of only if.



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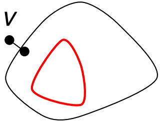
Proof: First, connected $\implies$ every vertex degree ≥ 1.
Sum of degrees is $2|V|-2$
Average degree $2-(2/|V|)$

Proof of only if.



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Proof: First, connected $\implies$ every vertex degree ≥ 1.
Sum of degrees is $2|V|-2$
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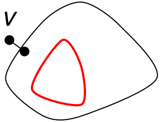
Not everyone is bigger than average! □

Proof of only if.



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By degree 1 removal lemma, $G-v$ is connected.

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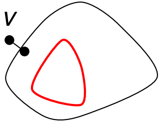

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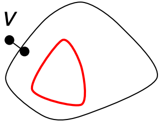

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Proof of $\implies$: By induction on $|V|$.
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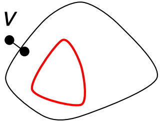
Induction Step: Assume for G with up to k vertices. Prove for $k + 1$
Claim: There is a degree 1 node.
Proof: First, connected $\implies$ every vertex degree ≥ 1.
Sum of degrees is $2|V| - 2$
Average degree $2 - (2/|V|)$
Not everyone is bigger than average! □
By degree 1 removal lemma, $G - v$ is connected.
$G - v$ has $|V| - 1$ vertices and $|V| - 2$ edges so by induction
$\implies$ no cycle in $G - v$.

Proof of only if.



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And no cycle in G since degree 1 cannot participate in cycle.

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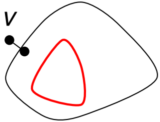

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Proof of “if part”

Thm:
“G is connected and has no cycles” $\implies$ “G connected and has $|V| - 1$ edges”
Proof:

Proof of “if part”

Thm:
“G is connected and has no cycles” $\implies$ “G connected and has $|V|-1$ edges”
Proof: Can we use the “degree 1” idea again?

Proof of “if part”

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“G is connected and has no cycles” $\implies$ “G connected and has $|V| - 1$ edges”
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Walk from a vertex using untraversed edges and vertices.

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Until get stuck. Why?

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Claim: Degree 1 vertex.

Proof of Claim:

Can’t visit more than once since no cycle.

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G has one more or $|V|-1$ edges.

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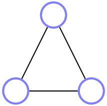
Planar graphs.

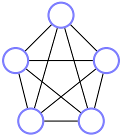
A graph that can be drawn in the plane without edge crossings.

Planar graphs.

A graph that can be drawn in the plane without edge crossings.



Planar?



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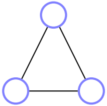

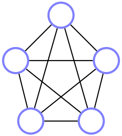

Planar? Yes for Triangle.

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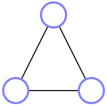

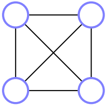

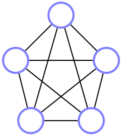

Planar? Yes for Triangle.
Four node complete K_4?

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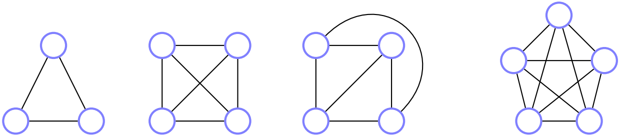

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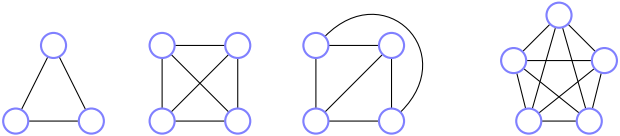

Planar? Yes for Triangle.
Four node complete K_4? Yes.
Five node complete or K_5 ?

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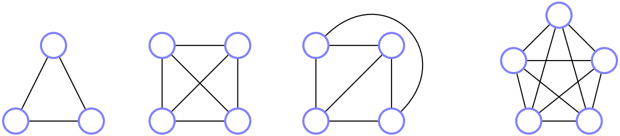

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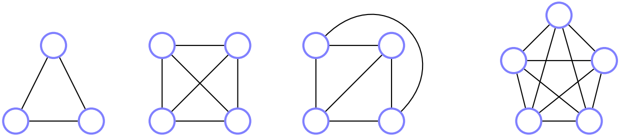

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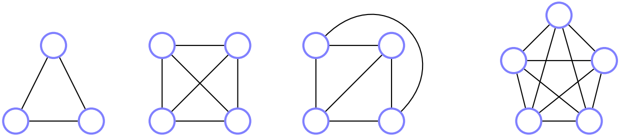

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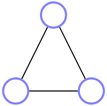
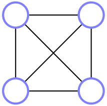
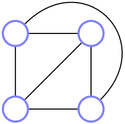
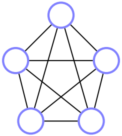

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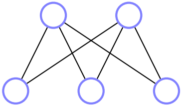

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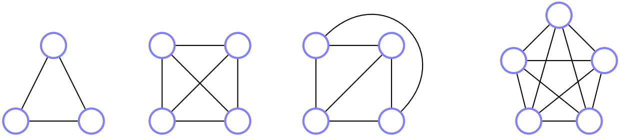

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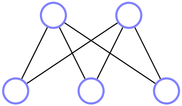
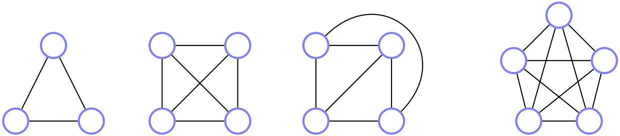


Two to three nodes, bipartite?

Planar graphs.

A graph that can be drawn in the plane without edge crossings.



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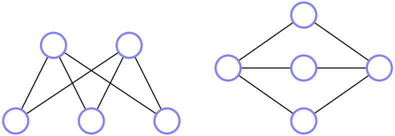

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Planar graphs.

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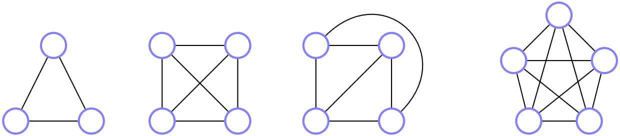

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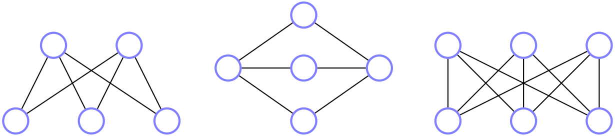
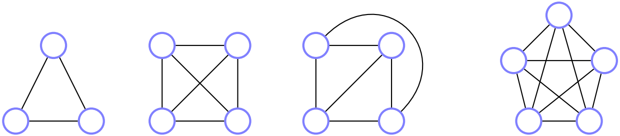


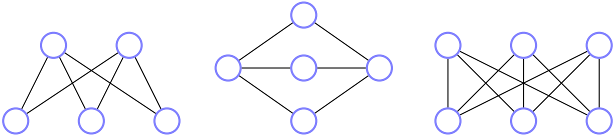
Two to three nodes, bipartite? Yes.
Three to three nodes, complete/bipartite or $K_{3,3}$.

Planar graphs.

A graph that can be drawn in the plane without edge crossings.



Planar? Yes for Triangle.
Four node complete K_4? Yes.
Five node complete or K_5 ? No! Why? Later.



Two to three nodes, bipartite? Yes.
Three to three nodes, complete/bipartite or $K_{3,3}$. No.

Planar graphs.

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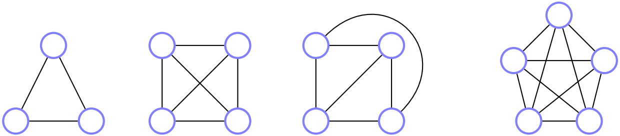

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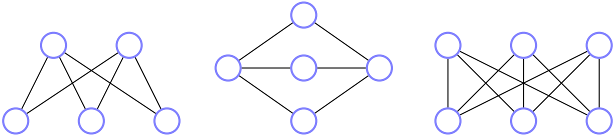

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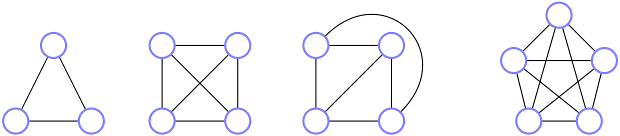

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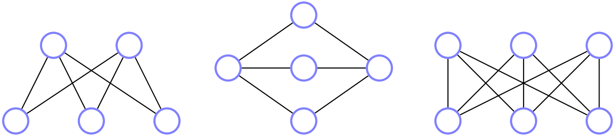

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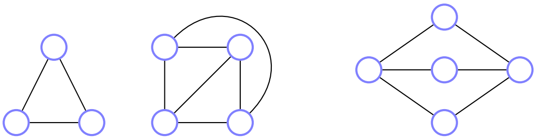

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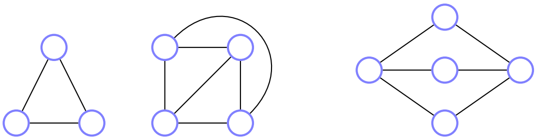

Faces: connected regions of the plane.

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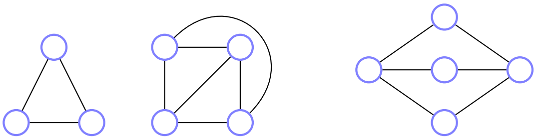

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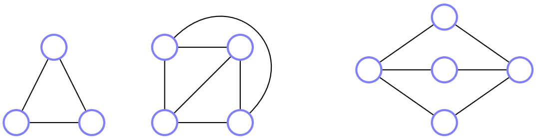

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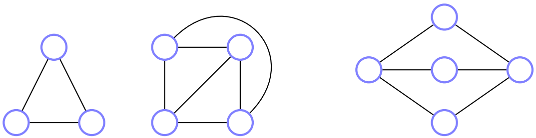

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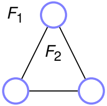

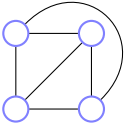
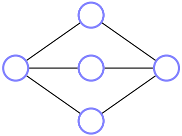

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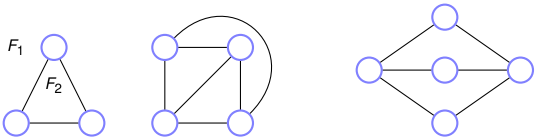

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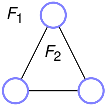

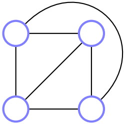
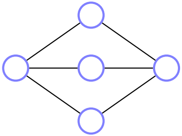

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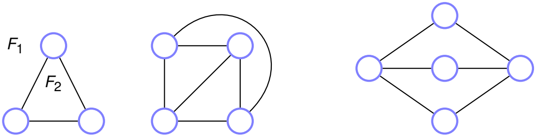

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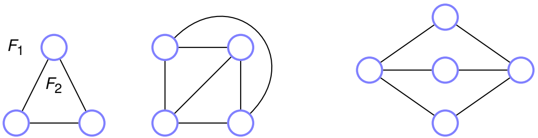

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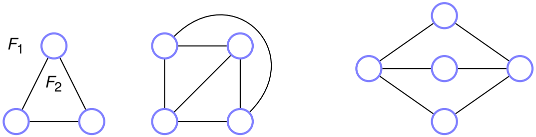

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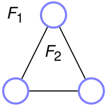

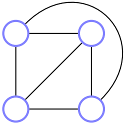

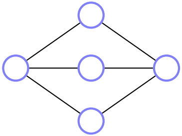

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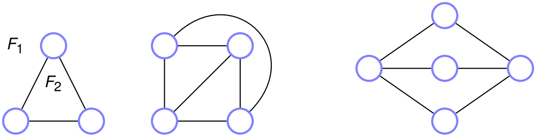

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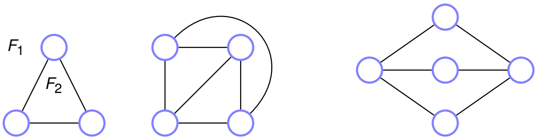

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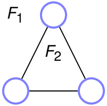

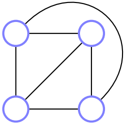

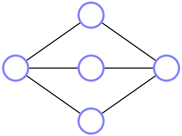

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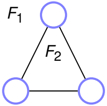

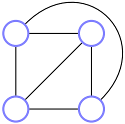

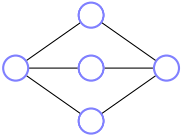

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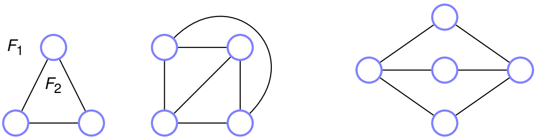

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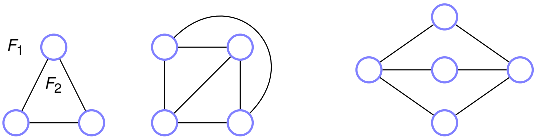

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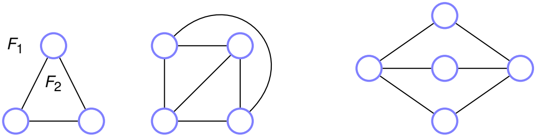

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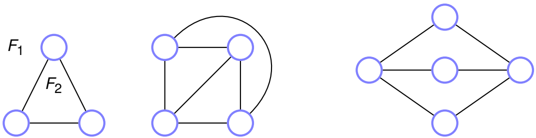

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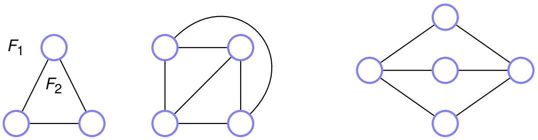

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Examples = 3!

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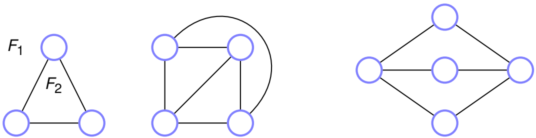

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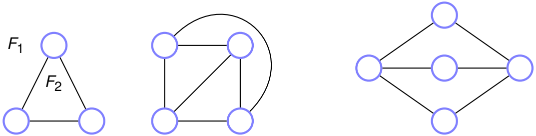

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Euler and Polyhedron.

Ancient Greek mathematicians knew formula for polyhedron.

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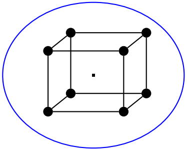

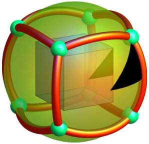

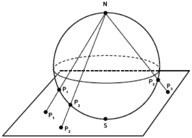

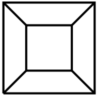

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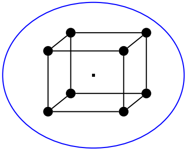
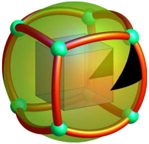
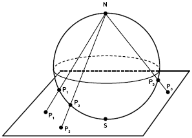

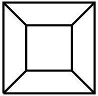

Faces?

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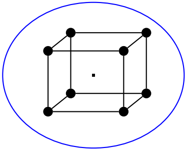

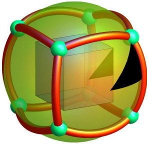

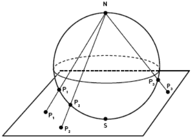

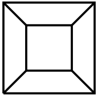

Faces? 6.   Edges?

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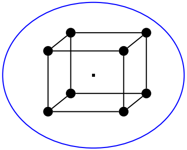
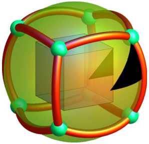
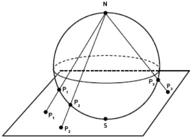
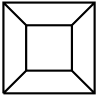




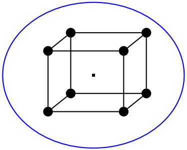


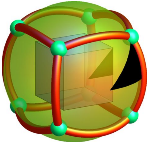


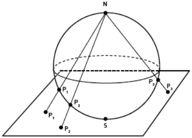
Faces? 6. Edges? 12.

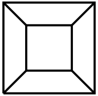
Euler and Polyhedron.

Ancient Greek mathematicians knew formula for polyhedron.









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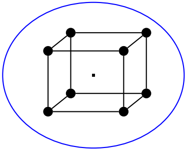

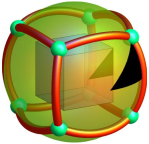

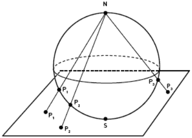

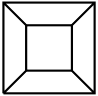

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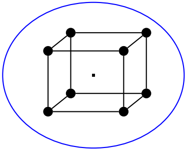
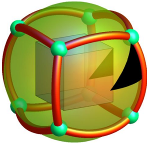
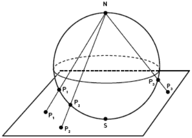

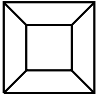

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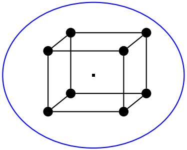
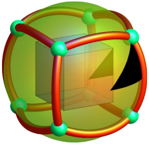
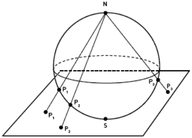

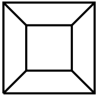

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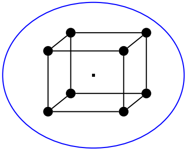

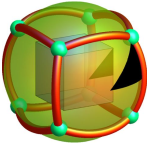

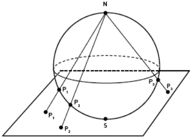

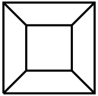

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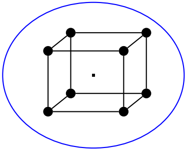

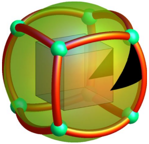

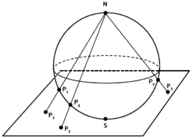

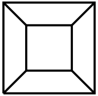

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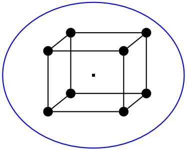

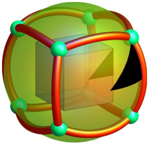

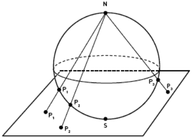

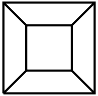

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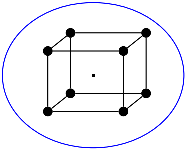
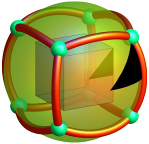
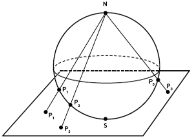
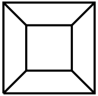

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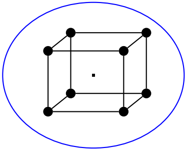

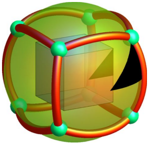

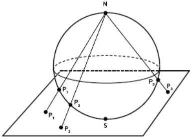

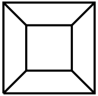

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Polyhedron without holes

Euler and Polyhedron.

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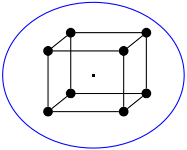 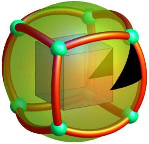 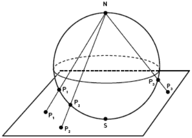 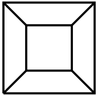

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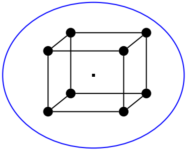

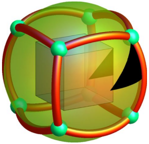

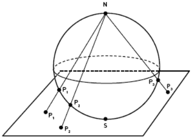

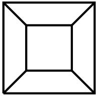

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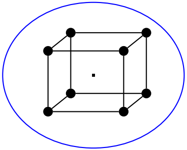

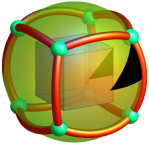

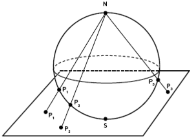

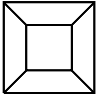

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Polyhedron without holes $\equiv$ Planar graphs.

For Convex Polyhedron:

Euler and Polyhedron.

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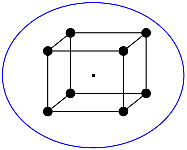

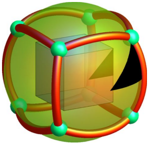

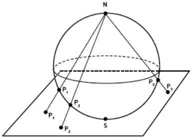

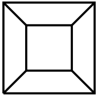

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For Convex Polyhedron:

Surround by sphere.

Euler and Polyhedron.

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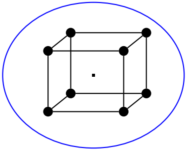

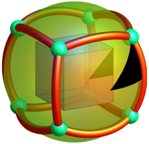

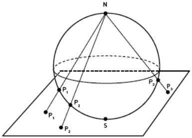

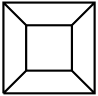

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For Convex Polyhedron:

Surround by sphere.

Project from internal point polytope to sphere:

Euler and Polyhedron.

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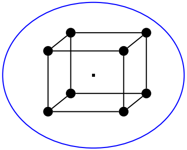

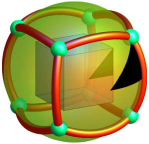

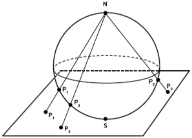

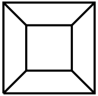

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For Convex Polyhedron:

Surround by sphere.

Project from internal point polytope to sphere: drawing on sphere.

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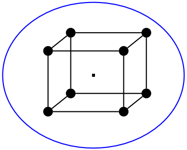

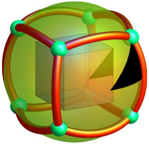

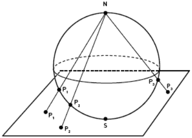

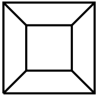

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For Convex Polyhedron:

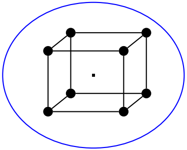
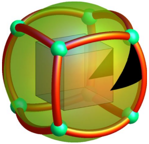
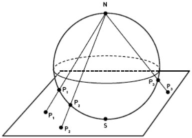
Surround by sphere.

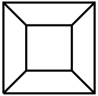
Project from internal point polytope to sphere: drawing on sphere.

Project Sphere-N

Euler and Polyhedron.

Ancient Greek mathematicians knew formula for polyhedron.



Faces? 6. Edges? 12. Vertices? 8.

Euler: Connected planar graph: $v + f = e + 2$.

$8 + 6 = 12 + 2$.

Greeks couldn't prove it. Induction? Remove vertex for polyhedron?

Polyhedron without holes $\equiv$ Planar graphs.

For Convex Polyhedron:

Surround by sphere.

Project from internal point polytope to sphere: drawing on sphere.

Project Sphere-N onto Plane:

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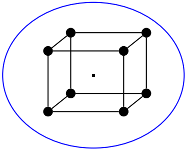

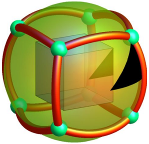

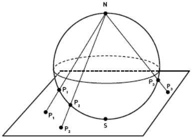

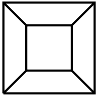

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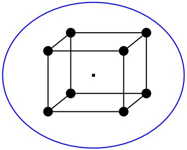
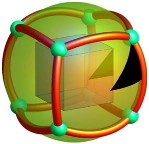
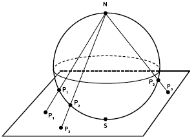
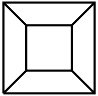

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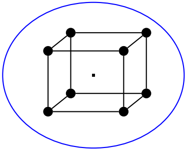

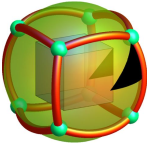

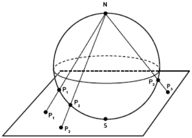

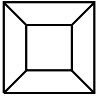

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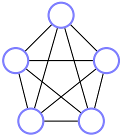
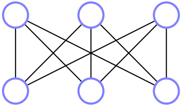
Euler proved formula thousands of years later!

Euler and non-planarity of K_5 and $K_{3,3}$





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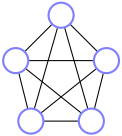

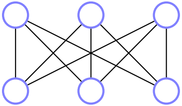

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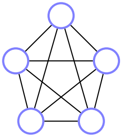

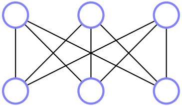

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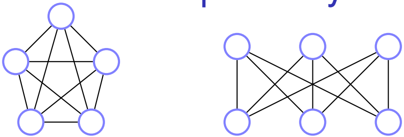

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Consider Face edge Adjacencies

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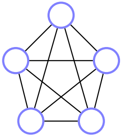

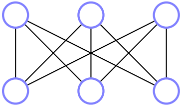

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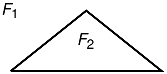

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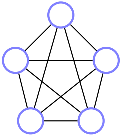

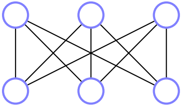

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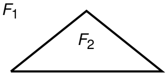

Each face is adjacent to at least three edges($v > 2$).

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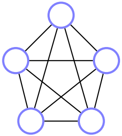

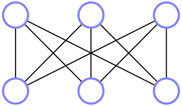

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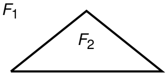

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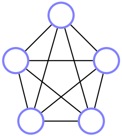

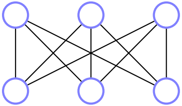

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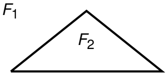

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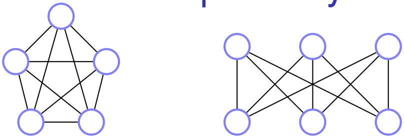

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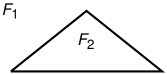

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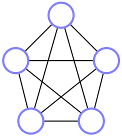

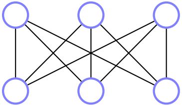

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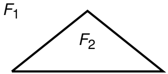

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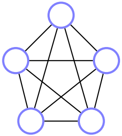

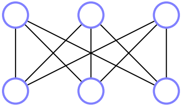

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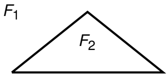

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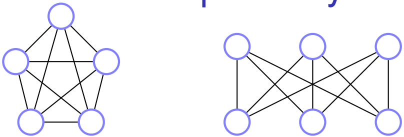

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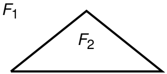

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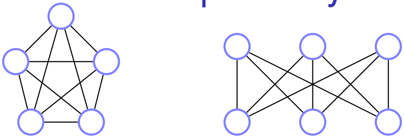

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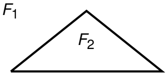

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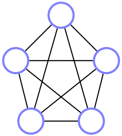

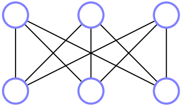

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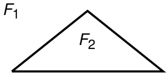

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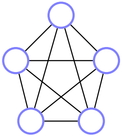

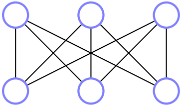

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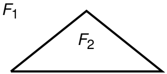

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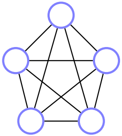
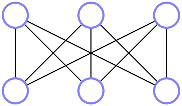
K_5

Euler and non-planarity of K_5 and $K_{3,3}$





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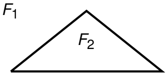

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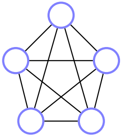
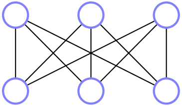
Plug into Euler: $v + \frac{2}{3}e \geq e+2 \implies e \leq 3v-6$

K_5 Edges?

Euler and non-planarity of K_5 and $K_{3,3}$

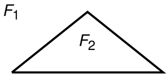




Euler: $v + f = e + 2$ for connected planar graph.

We consider simple graphs where $v \geq 3$.

Consider Face edge Adjacencies



Each face is adjacent to at least three edges($v > 2$).

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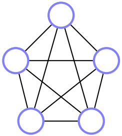
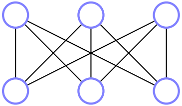
Euler and non-planarity of K_5 and $K_{3,3}$





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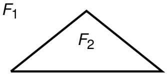

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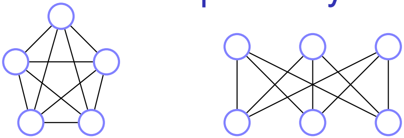
Euler and non-planarity of K_5 and $K_{3,3}$



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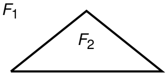

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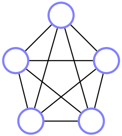
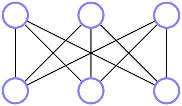
Euler and non-planarity of K_5 and $K_{3,3}$





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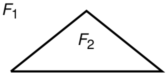

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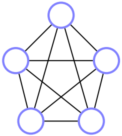
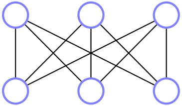
Euler and non-planarity of K_5 and $K_{3,3}$





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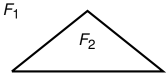

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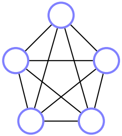
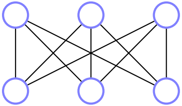
K_5 Edges? $e = 4 + 3 + 2 + 1 = 10$. Vertices? $v = 5$.
$10 \not\leq 3(5) - 6 = 9$.

Euler and non-planarity of K_5 and $K_{3,3}$





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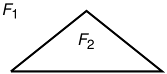

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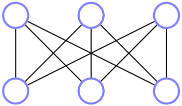
K_5 Edges? $e = 4 + 3 + 2 + 1 = 10$. Vertices? $v = 5$.

$10 \not\leq 3(5) - 6 = 9$. $\implies$ K_5 is not planar.

Proving non-planarity for $K_{3,3}$



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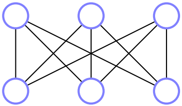

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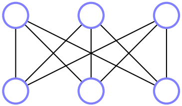

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$e \leq 3(v) - 6$ for planar graphs.

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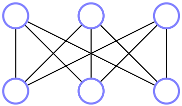

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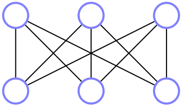

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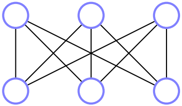

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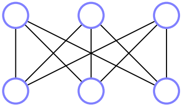
Need a different approach!

Proving non-planarity for $K_{3,3}$



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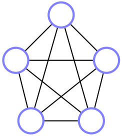
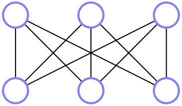
Need a different approach! See notes for details.

Summary: Planarity and Euler





These graphs **cannot** be drawn in the plane without edge crossings.

Proof of Euler's formula.

Theorem (Euler): Connected planar graph has $v + f = e + 2$.

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Theorem (Euler): Connected planar graph has $v + f = e + 2$.

Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.

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Proof: Induction on e.
Base: $e=0$, $v=f=1$.
Induction Step:

Proof of Euler's formula.

Theorem (Euler): Connected planar graph has $v + f = e + 2$.

Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.
Induction Step:
First, if it is a tree:

Proof of Euler's formula.

Theorem (Euler): Connected planar graph has $v+f=e+2$.

Proof: Induction on e.
Base: $e=0$, $v=f=1$.
Induction Step:
First, if it is a tree: $e=v-1$, $f=1$, $v+1=(v-1)+2$. Done.

Proof of Euler's formula.

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Proof: Induction on e.
Base: $e=0$, $v=f=1$.
Induction Step:
First, if it is a tree: $e=v-1$, $f=1$, $v+1=(v-1)+2$. Done.
Suppose it is NOT a tree: Assume holds for $e \leq n$. Consider $e=n+1$.

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Find a cycle.

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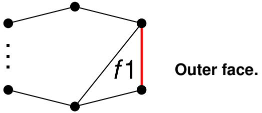

Joins two faces.

Proof of Euler's formula.

Theorem (Euler): Connected planar graph has $v + f = e + 2$.

Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.
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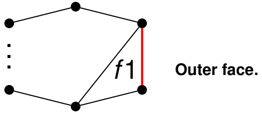

Joins two faces.
New graph: v-vertices.

Proof of Euler's formula.

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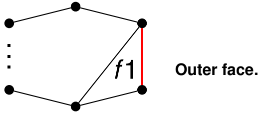

Joins two faces.
New graph: v-vertices. $e-1$ edges.

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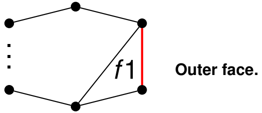

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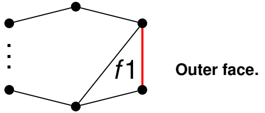

Joins two faces.
New graph: v-vertices. $e - 1$ edges. $f - 1$ faces. Planar.

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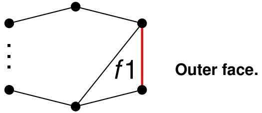

Joins two faces.
New graph: v-vertices. $e - 1$ edges. $f - 1$ faces. Planar.
$v + (f - 1) = (e - 1) + 2$ by induction hypothesis.

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Proof: Induction on e.
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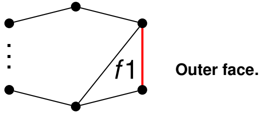

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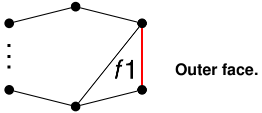

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Hypercubes.

Recall:

Complete graphs, really well connected!

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$|V|(|V|-1)/2$

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Trees, connected, few edges.

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Hypercubes. Well connected. $|V|\log|V|$ edges!

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Also represents bit-strings nicely.

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A hypercube is a graph $G=(V,E)$

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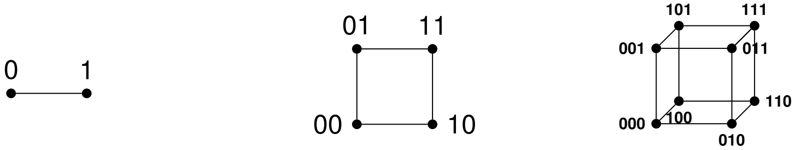

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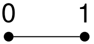

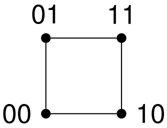

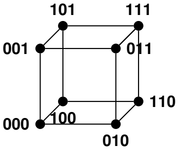

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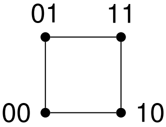

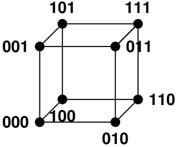

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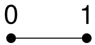

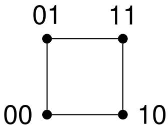

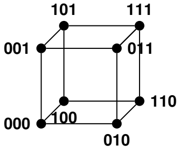

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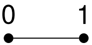

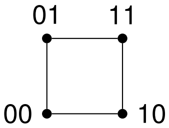

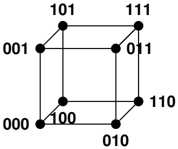

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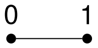

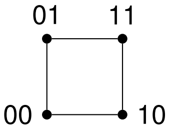

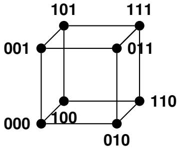

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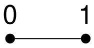

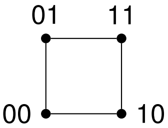

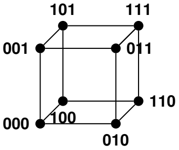

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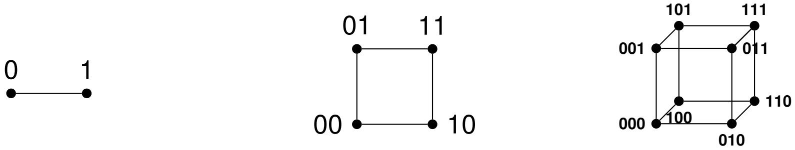

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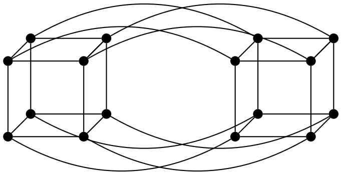

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Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

Proof of Large Cuts.

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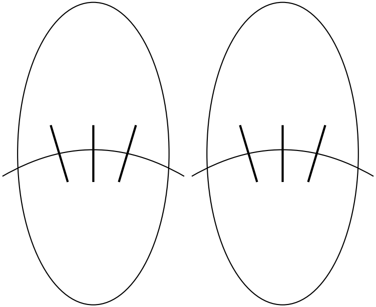

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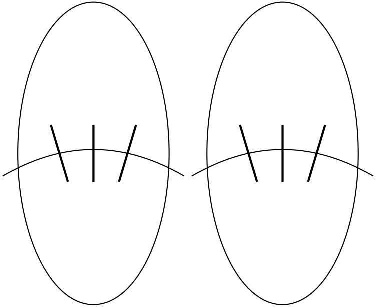

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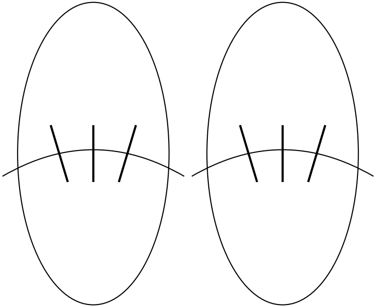

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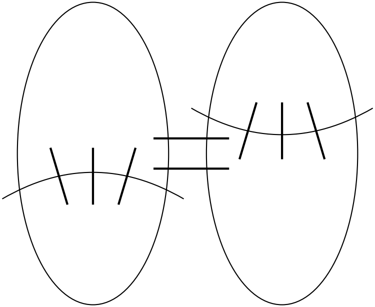

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Edges cut in $H_0 \geq |S_0|$.
Edges cut in $H_1 \geq |S_1|$.

Total cut edges $\geq |S_0| + |S_1| = |S|$. □

Induction Step. Case 2.

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.
Proof: Induction Step. Case 2. $|S_0| \geq |V_0|/2$.

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Recall Case 1: $|S_0|, |S_1| \leq |V|/2$

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Edges in E_x connect corresponding nodes.

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Total edges cut:

Induction Step. Case 2.

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$|V_0|$

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$|V_0| = |V|/2 \geq |S|$. □

Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

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