Lecture 7 Outline.

1. Modular Arithmetic.
   Clock Math!!!
2. Inverses for Modular Arithmetic: Greatest Common Divisor (GCD).
3. Euclid's GCD Algorithm

Clock Math

If it is 4:00 now.

What time is it in 5 hours? 9:00!

What time is it in 15 hours? 19:00!

Actually 7:00.

19 is the “same as 7” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 104:00! or 8:00.

8 is the same as 104 for a 12 hour clock system.

Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in $\{1, \ldots, 11, 12\}$

Day of the week.

Today is Tuesday.
What day is it a year from now? on February 6, 2025?
Number days.
0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 2.
4 days from now. day 6 or Saturday.
24 days from now. day 26 or day 5, which is Friday!
two days are equivalent up to addition/subtraction of multiple of 7.
10 days from now is day 5 again, Friday!

What day is it a year from now?
This year is a leap year! So 366 days from now.
Day 2+366 or day 368.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
368/7 leaves quotient of 52 and remainder 4.
or February 6, 2025 is Day 4, a Thursday.

Years and years...

80 years from now? February 6, 2104
  20 leap years. 366*20 days
  60 regular years. 365*60 days
It is day $2 + 366 * 20 + 365 * 60$. Equivalent to?

Hmm.
  What is remainder of 366 when dividing by 7?   2.
  What is remainder of 365 when dividing by 7?   1

Today is day 2.
    Get Day: 2 + 20*2 + 60*1 = 102
    Remainder when dividing by 7? 4.
    Or February 6, 2104 is Thursday!

Further Simplify Calculation:
   20 has remainder 6 when divided by 7.
   60 has remainder 4 when divided by 7.
Get Day: 2 + 6*2 + 4*1 = 18.
  Or Day 4.   February 6, 2104 is Thursday.

“Reduce” at any time in calculation!

Modular Arithmetic: Basics.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x-y)$ is divisible by m.
...or $x = y + km$ for some integer k.
...or x and y have the same remainder w.r.t. m.

Mod 7 equivalence classes:
$\{\ldots,-7,0,7,14,\ldots\}$ $\{\ldots,-6,1,8,15,\ldots\}$ ...

Useful Fact: Addition, subtraction, multiplication can be done with any equivalent x and y.

or " $a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$
$\implies a + b \equiv c + d \pmod{m}$ and $a \cdot b = c \cdot d \pmod{m}$"

Proof: If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer k.
If $b \equiv d \pmod{m}$, then $b = d + jm$ for some integer j.
Therefore, $a + b = c + d + (k + j)m$ and since $k + j$ is integer.
$\implies a + b \equiv c + d \pmod{m}$. $\square$

Can calculate with representative in $\{0,\ldots,m-1\}$.

Notation

$x \pmod m$ or $\mod(x, m)$- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x,m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29,12) = 29 - (\lfloor \frac{29}{12} \rfloor) * 12 = 29 - (2) * 12 = 5$

Recap:
$a \equiv b \pmod m$.
Says two integers a and b are equivalent modulo m.

Modulus is m

Inverses and Factors.

Division: multiply by multiplicative inverse.

$$2x = 3 \implies (1/2)\cdot 2x = (1/2)3 \implies x = 3/2.$$

Multiplicative inverse of x is y where $xy = 1$;
1 **is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element.

Multiplicative inverse of x **mod** m is y with $xy = 1 \pmod{m}$.

For 4 modulo 7 inverse is 2: $2 \cdot 4 \equiv 8 \equiv 1 \pmod{7}$.

Can solve $4x = 5 \pmod{7}$.

$2 \cdot 4x = 2 \cdot 5 \pmod{7}$

$8x = 10 \pmod{7}$

$x = 3 \pmod{7}$

Check! $4(3) = 12 = 5 \pmod{7}$.

$x = 3 \pmod{7}$ ::: Check! $4(3) = 12 = 5 \pmod{7}$.

For 8 modulo 12: no multiplicative inverse!

"Common factor of 4" $\implies$

$8k - 12\ell$ is a multiple of four for any ℓ and k $\implies$

$8k \not\equiv 1 \pmod{12}$ for any k.

Greatest Common Divisor and Inverses.

Thm:
If greatest common divisor of x and m, $\gcd(x,m)$, is 1, then x has a multiplicative inverse modulo m.

Proof $\implies$: The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo m.

Pigenhole principle: Each of m numbers in S correspond to different one of m equivalence classes modulo m.
$\implies$ One must correspond to 1 modulo m.

If not distinct, then $a, b \in \{0, \ldots, m-1\}$, where
$(ax \equiv bx \pmod{m}) \implies (a-b)x \equiv 0 \pmod{m}$
Or $(a-b)x = km$ for some integer k.

$gcd(x,m) = 1$
$\implies$ Prime factorization of m and x do not contain common primes.
$\implies$ $(a-b)$ factorization contains all primes in m's factorization.
So $(a-b)$ has to be multiple of m.
$\implies$ $(a-b) \geq m$. But $a, b \in \{0, ...m-1\}$. Contradiction. $\square$

Proof review. Consequence.

Thm: If $\gcd(x,m) = 1$, then x has a multiplicative inverse modulo m.

Proof Sketch: The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo m.

$\square$

...

For $x = 4$ and $m = 6$. All products of 4...

$S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}$

reducing $\pmod 6$

$S = \{0, 4, 2, 0, 4, 2\}$

Not distinct. Common factor 2.

For $x = 5$ and $m = 6$.

$S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}$

All distinct, contains 1! 5 is multiplicative inverse of 5 $\pmod 6$.

$5x = 3 \pmod 6$ What is x? Multiply both sides by 5.

$x = 15 = 3 \pmod 6$

$4x = 3 \pmod 6$ No solutions. Can't get an odd.

$4x = 2 \pmod 6$ Two solutions! $x = 2, 5 \pmod 6$

Very different for elements with inverses.

Finding inverses.

How to find the inverse?

How to find **if** x has an inverse modulo m?

Find gcd (x, m).
  Greater than 1? No multiplicative inverse.
  Equal to 1? Mutliplicative inverse.

Algorithm: Try all numbers up to x to see if it divides both x and m.

Very slow.

Next: A Faster algorithm.