Public-Key Cryptography

1. Cryptography $\Rightarrow$ relation to Bijections
2. Public-Key Cryptography
3. RSA system
   3.1 Efficiency: Repeated Squaring.
   3.2 Correctness: Fermat's Little Theorem.
   3.3 Construction.

Cryptography ...

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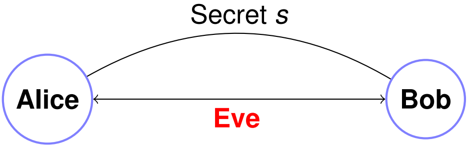

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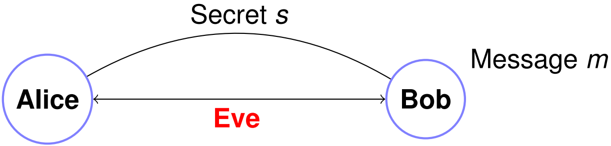

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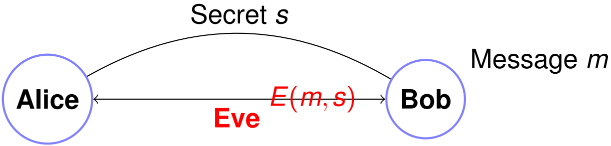

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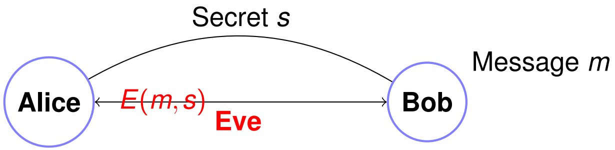

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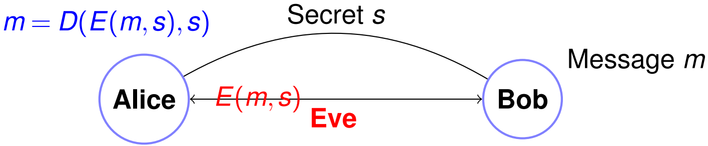

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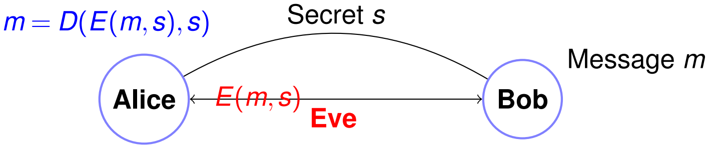

What is the relation between D and E (for the same secret s)?

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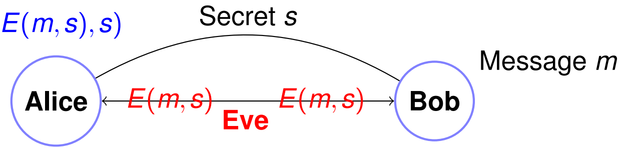

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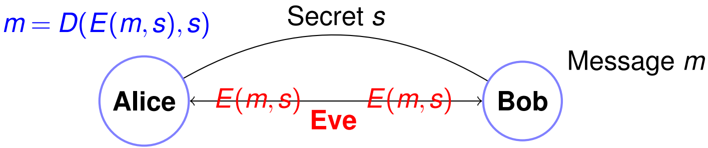

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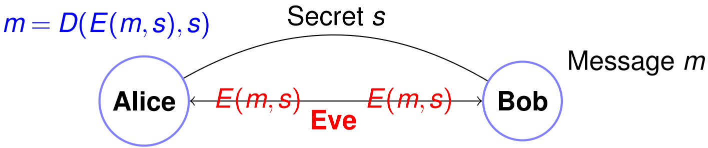

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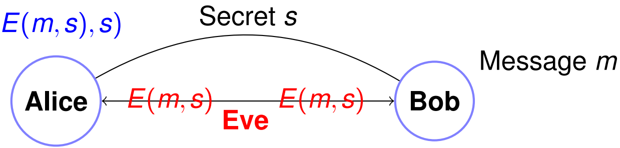

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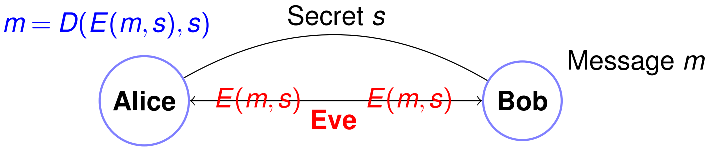

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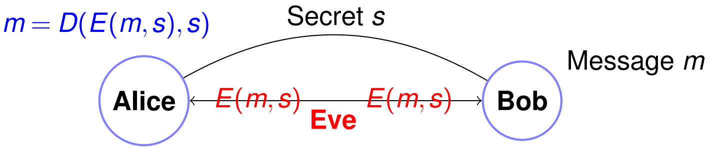

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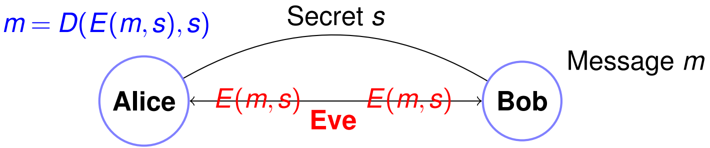

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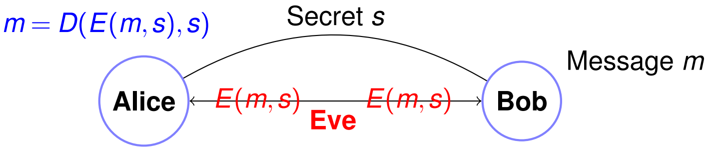

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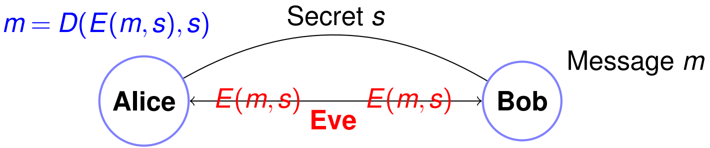

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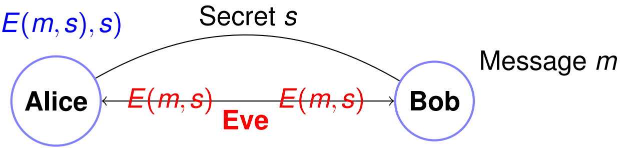

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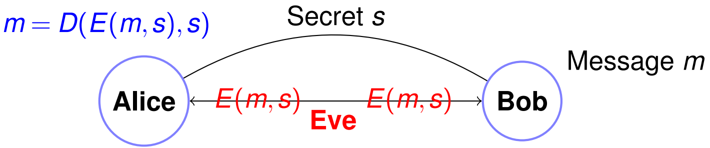

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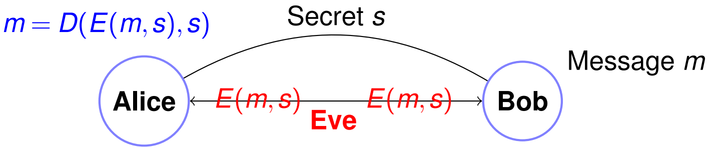

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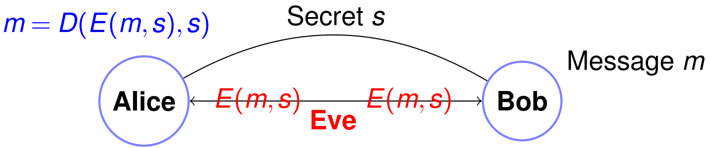

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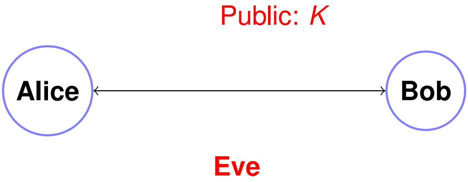

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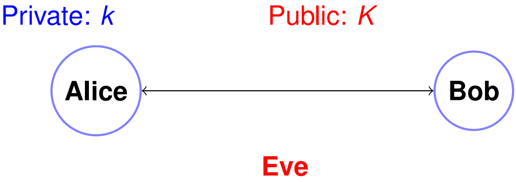

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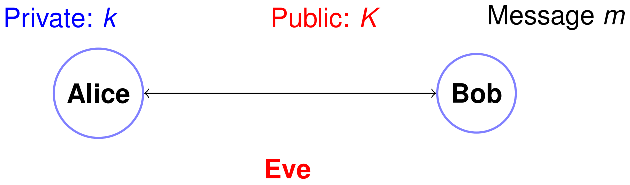

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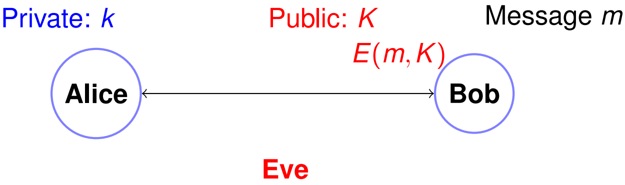

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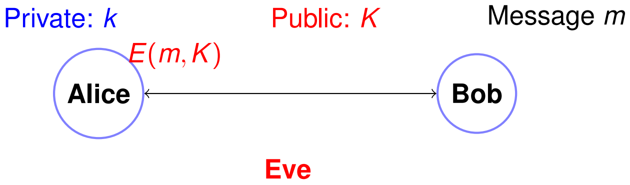

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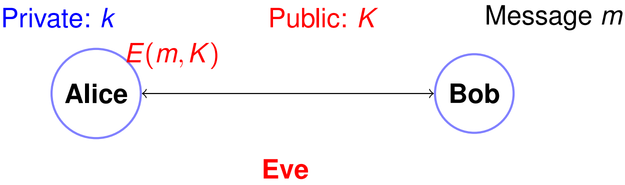

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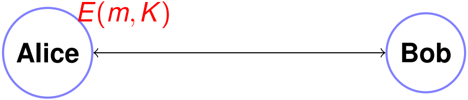
Private: k Public: K Message m

$E(m,K)$



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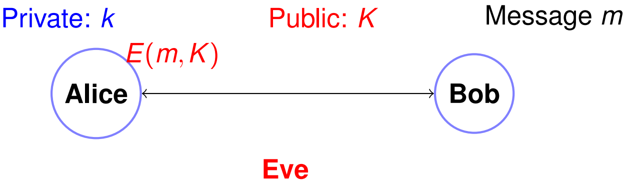

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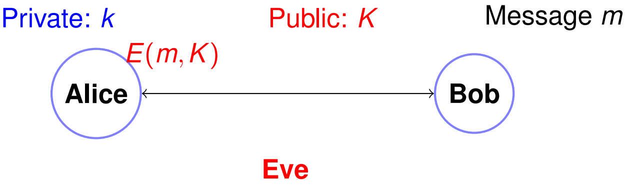

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Public: K

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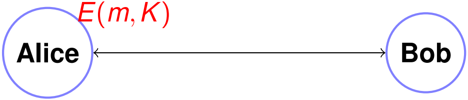

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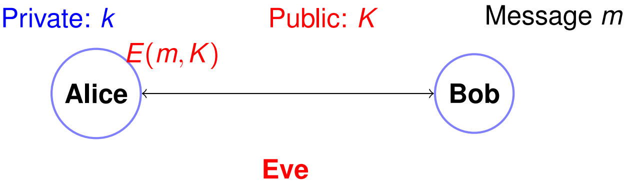

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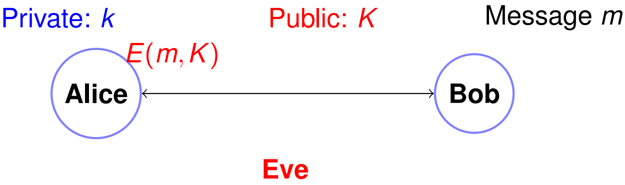

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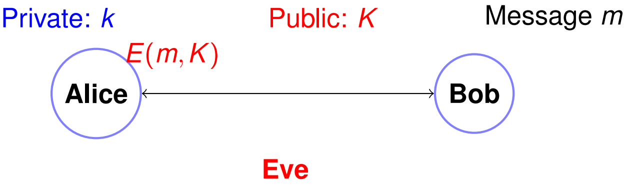

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(Only?) Alice can decode with k.

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We don't really know.

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Decoding: $D(y)$ is $\mod (y^d, N)$.

Does $D(E(m)) = m^{ed} = m \mod N$?

Yes!

---

[1] Typically small, say $e = 3$.

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$d = e^{-1} = -17 = 43 = \pmod{60}$

Important Considerations

Q1: Why does RSA work correctly?

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Q1: Why does RSA work correctly? Fermat's Little Theorem!

Q2: Can RSA be implemented efficiently? Yes, repeated squaring!

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$E(2)$

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Repeated squaring $O(\log y)$ multiplications versus y!!!

1. x^y: Compute x^1,

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1. x^y: Compute $x^1, x^2, x^4, \ldots, x^{2^{\lfloor \log y \rfloor}}$.

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Repeated squaring $O(\log y)$ multiplications versus y!!!

1. x^y: Compute $x^1, x^2, x^4, \ldots, x^{2^{\lfloor \log y \rfloor}}$.
2. Multiply together x^i where the $(\log(i))$th bit of y is 1.

Always decode correctly?

Fermat's Little Theorem: For prime p, and $a \not\equiv 0 \pmod{p}$,
$a^{p-1} \equiv 1 \pmod{p}$.

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$$(a \cdot 1) \cdot (a \cdot 2) \cdots (a \cdot (p-1)) \equiv 1 \cdot 2 \cdots (p-1) \mod p,$$

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Since multiplication is commutative.

$$a^{(p-1)}(1 \cdots (p-1)) \equiv (1 \cdots (p-1)) \mod p.$$

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$\square$

RSA and Fermat: mathematical connection

Thm: $m^{ed} = m \pmod{pq}$ if $ed = 1 \pmod{(p-1)(q-1)}$

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$3^{19} \pmod{7}$?

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$3^{19} \pmod{7}$? $3^{3*6+1} \pmod{7}$? $(3^{3*6} * 3) \pmod{7}$? 3.

Corollary: $a^{k(p-1)+1} = a \pmod{p}$

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All steps are polynomial in $O(\log N)$, the number of bits.

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3. Breaking this scheme $\implies$ factoring N.

Security of RSA.

Security?

1. Alice knows p and q (and d, and other numbers).
2. Bob only knows, $N(= pq)$, and e.

   Does not know, for example, d or factorization of N.
3. Breaking this scheme $\implies$ factoring N.
   Don't know how to factor N efficiently on regular computers.

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Further study: CS161 and CS171.